

Triangle Tiling III: The Triquadratic Tilings

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Abstract

An N -tiling of triangle ABC by triangle T (the “tile”) is a way of writing ABC as a union of N copies of T overlapping only at their boundaries. We continue the work of [1, 2] on classifying the possible triangle tilings. The results of all three papers together enable us to make some progress on two problems of Erdős about triangle tilings, one of which asks, “for which N is there an N -tiling of some triangle by some tile?”. This paper takes up a case that was left unsolved in [1, 2], which is the following. Let the tile T have angles α , β , and γ , and suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π . Suppose there is an N -tiling of ABC . Then we classify the triples (ABC, T, N) such that there is an N -tiling of ABC by T . Our solution is as follows: There is an N -tiling of ABC by T if and only if N is a square times a product of distinct primes of the form $8n \pm 1$, or 2, and the “tiling equation” $M^2 + N = 2K^2$ is solvable in positive integers M and K with $M^2 < N$ and K divides M^2 . In that case the tile can be taken to have sides a , b , and c where $a = M$ and $c = K$ and $b = K - M^2/K$, so all three sides will be integers.

Combining this with the results of [2], we obtain a no-tiling theorem: if N is squarefree and $N > 6$ and divisible by at least one prime congruent to 3 mod 4, then there are no N -tilings of any triangle by any tile, unless the tile has a 120° angle.

1 Introduction

This paper is the third in a series of papers on Triangle Tiling [1, 2]. An N -tiling of triangle ABC by triangle T (the “tile”) is a way of writing ABC as a union of N copies of T overlapping only at their boundaries. The general aim of this work is to understand the nature of triangle tilings. We can exhibit quite a few families of triangle tilings, some of which are very familiar, but at least two completely new families of tilings have been discovered in the course of this work. We would like to think that every triangle tiling belongs to one of these families. This we cannot prove, since even for the triangles ABC and tiles T for which we know some tilings, we cannot prove that there are no *other* tilings. What we have accomplished is to completely understand the triples (ABC, T, N) such that there exists a tiling of ABC by T using N tiles. In particular, this understanding enables us to answer more specific questions, such as, for which N does there exist a triangle ABC and a tile T and an N -tiling of ABC by T ?

Cases of specific N were quite interesting, before we developed our general theory. The question that first sparked our interest was whether any triangle can be 7-tiled. We gave a long Euclidean-style proof that no 7-tiling exists, but it was clear that a similar proof for 11-tilings might be a thousand pages long. Now we know that there is also no 11-tiling. Indeed, there is no N -tiling for $N = 7, 14$, or 23, each of which we proved individually at the outset of this work. But there *does* exist a 28-tiling, which we think has never been seen by human eyes before. We generalize the 28-tiling, showing it to be the simplest member of a new family of tilings. These are the “triquadratic

tilings”; they exist when K divides N , or equivalently K divides M^2 . On the other hand, we prove that when K does not divide M^2 , no N -tiling exists with the tile determined by N and K . There are, for example, no N -tilings for $N = 31, 41, 63$, etc.

In [1], we used techniques of linear algebra and field theory to completely settle the following cases: When T is similar to ABC ; when T is a right triangle; when T is isosceles; and when ABC is isosceles. In [2], we made a frontal assault on the general case, when T is not similar to ABC and T is not a right triangle. A number of special cases (special values of α and β , or case in which α and β satisfy a certain equation) required special treatment. Many of these special cases were treated using field theory or algebraic number theory. But if the sides a , b , and c are commensurate (have rational ratios), then we might as well assume they are integers, and field theory is useless, because the field $\mathbb{Q}(a, b, c)$ and its subfields all collapse to \mathbb{Q} . There were then two cases left unsolved in [2]. The first of these is taken up in this paper. The second is taken up in [3].

Let the angles of the tile opposite sides a , b , and c be α , β , and γ . We show a , b , and c are commensurate if and only if $\sin(\alpha/2)$ is rational, and that in that case, α is not a rational multiple of π , except for two specific values of α , which have already been treated in [2], where it is shown that no tilings use those tiles. Hence, in most of this paper, we can assume that α is not a rational multiple of π .

Between the three numbers α , β , and γ , we have two linear relations. The first one is $\alpha + \beta + \gamma = \pi$, because they are the three angles of a triangle. The second relation arises from the “vertex splitting” at the vertices of the tiled triangle ABC . Consider all the tiles that share any of the three vertices, some (or none) have α angles at the vertex, some (or none) have β angles, and some (or none) have γ angles. Thus for some nonnegative integers P , Q , and R , we have

$$P\alpha + Q\beta + R\gamma = \pi.$$

If $P = Q = R = 1$ then ABC is similar to T ; but that case was treated in [1] using linear algebra, and in this paper we assume T is not similar to ABC . Hence the two linear relations are independent relations; substituting $\gamma = \pi - \alpha - \beta$ we have a non-trivial linear relation between α and β . If we could get a third independent linear relation between the angles, we could solve for α , β and γ . At each vertex of the tiling, some angles add up to either π (for a boundary vertex, or a “non-strict” interior vertex, i.e. one lying on a side of another tile), or to 2π , for a strict interior vertex. Because α is not a rational multiple of π , it is not a rational multiple of β either; so it will never be the case that the same angle can be composed of some α angles and also of some β angles. Hence the possibilities for angles meeting at interior vertices can be controlled to some extent. In [2] we carry out this analysis in detail; this analysis gives rise to a number of special cases (when we get another relation and can solve for α and β), which are treated in [2]. But there is one possibility in which, despite the techniques that succeed in all other cases in [2], we cannot get another relation. That is the case when $P = 3$ and $Q = 2$, i.e., the vertices of ABC are together composed of three α and two β angles, and so $3\alpha + 2\beta = \pi$, and $\gamma = \beta + 2\alpha$. We can show that in that case, the vertex angles of ABC must be 2α , β , and γ . In this paper, unlike the others in this series, we do not assume $\alpha < \beta$; otherwise there would also be the case $2\alpha + 3\beta = \pi$. We do still assume α and β are less than γ , and $\alpha \neq \beta$.

Therefore, the basic assumptions for this paper are

- $3\alpha + 2\beta = \pi$.
- α is not a rational multiple of π .
- $\sin(\alpha/2)$ is rational, or (equivalently) a , b , and c are commensurate.

- Triangle ABC has angle 2α at A , β at B , and $\alpha + \beta$ at C , and two tiles meet at vertex A , and two at vertex C .

Our first result is:

(1) If there is an N -tiling of ABC by T under these assumptions, then N is a square times a product of distinct primes of the form $8n \pm 1$ or 2, and the “tiling equation” $M^2 + N = K^2$ (with $M^2 < N$) is solvable.

This equation has only finitely many solutions for each N ; and it follows that

For a given N satisfying these assumptions, one can explicitly compute a finite set of possible tiles (one for each solution of the tiling equation) and for each tile, just one triangle ABC , such that if any N -tiling exists, it uses one of those tiles and tiles the corresponding ABC .

Originally we thought there were no N -tilings of any triangle by any tile under these assumptions; and we verified this by hand for $N = 7, 14, 17$, and 23. It was too difficult to check $N = 28$, so we wrote a computer program to check it. To our surprise, we discovered a 28-tiling! That led to the discovery of the second main result of this paper:

(2) If the tiling equation is solvable and K divides N (or equivalently K divides M^2), then there does exist a tiling. These are the “triquadratic tilings.” These seem to be completely new.

The simplest triquadratic tiling is for $N = 28$. Then we complement our existence theorem by a non-existence theorem:

(3) If N satisfies the condition in (1) but not the condition in (2) then there are no N -tilings of any triangle by any tile.

The principal tool in the proof of (3) is the introduction of a graph \mathbb{H} whose nodes are the tiles of a tiling, in which two nodes are immediately connected if they share two vertices but have different angles at the shared vertices. Analysis of the connected components of this graph helps us to control the possible partial tilings.

2 Four lemmas from previous work

In this section we review some results from [1] and [2], so that this paper can be more or less self-contained.

The \mathbf{d} matrix is defined by

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = XYZ$$

where X is the length of BC (opposite the 2α angle), Y is the length of AC (opposite the β angle) and Z is the length of AB (opposite the β angle). We always draw our pictures with BC horizontal and A at the lower left, or southwest. The \mathbf{d} matrix tells how the sides of ABC are composed of edges of tiles (although it contains no information about the order of those edges). In case the tile is similar to ABC , then there is an eigenvalue equation

$$\mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

since $(X, Y, Z) = \lambda(a, b, c)$ in that case. In present case we can make the equation look something like that:

The following lemma was proved in [2].

Lemma 1 *Let X , Y , and Z be the lengths of BC , AC , and AB respectively. Let $\lambda = K$. Let $t = 2 - s^2$, where $s = 2 \sin(\alpha/2)$. Then*

$$\mathbf{d} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} ta \\ b \\ c \end{pmatrix}$$

Because of the factor t , this is not quite an eigenvalue problem. When triangle ABC is similar to the tile, the factor t does not appear, but still we can make this problem look *almost* like an eigenvalue problem.

The quantity $s = 2 \sin(\alpha/2)$ was used extensively in [2]. It is useful because the ratios a/c and b/c can be expressed simply in terms of s , as shown in the following lemma, whose short and useful proof we repeat here.

Lemma 2 *Suppose $3\alpha + 2\beta = \pi$, and $s = 2 \sin \alpha/2$. Then we have (whether or not s is rational)*

$$\begin{aligned} \sin \gamma &= \cos \frac{\alpha}{2} \\ \frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2 \end{aligned}$$

Proof. Since $\gamma = \pi - (\alpha + \beta)$, we have

$$\begin{aligned} \sin \gamma &= \sin(\pi - (\alpha + \beta)) \\ &= \sin(\alpha + \beta) \\ &= \cos(\pi/2 - (\alpha + \beta)) \\ &= \cos \frac{\alpha}{2} \quad \text{since } \pi/2 - \beta = 3\alpha/2 \end{aligned}$$

Then $c = \sin \gamma = \cos \alpha/2$, and $a = \sin \alpha = 2 \sin(\alpha/2) \cos(\alpha/2)$. Hence

$$\frac{a}{c} = 2 \sin \alpha/2.$$

Since $3\alpha + 2\beta = \pi$, we have

$$\begin{aligned} \sin \beta &= \sin(\pi/2 - 3\alpha/2) \\ &= \cos(3\alpha/2) \\ &= 4 \cos^3 \frac{\alpha}{2} - 3 \cos \frac{\alpha}{2} \end{aligned}$$

Hence

$$\begin{aligned} b/c &= 4 \cos^2(\alpha/2) - 3 \\ &= 4(1 - \sin^2 \alpha/2) - 3 \\ &= 1 - 4 \sin^2 \alpha/2 \end{aligned}$$

Then we have

$$\begin{aligned}\frac{a}{c} &= s \\ \frac{b}{c} &= 1 - s^2\end{aligned}$$

That completes the proof of the lemma.

The following lemma, also proved in [2], has a difficult computational proof, too long to repeat here. But we need the result, so we at least restate the lemma here.

Lemma 3 *Suppose ABC has angles 2α , β , and $\beta + \alpha$ and $\sin(\alpha/2)$ is irrational. Then $s = 2\sin(\alpha/2)$ satisfies a cubic equation over \mathbb{Q} ; hence its degree over \mathbb{Q} is 3.*

Proof. See [2].

In particular if $s = 2\sin(\alpha/2)$ satisfies a quadratic equation, then it is rational. That is the form in which we shall use this lemma.

Finally we state this lemma, also proved in [2]

Lemma 4 *Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π , and there is an N -tiling of ABC by the tile with angles α and β . Then triangle ABC has angles 2α , β , and $\alpha + \beta$.*

Proof. See [2].

3 Types of vertices

Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of $\pi/2$. The “type” of a vertex V in a tiling is a triple (n, k, j) , meaning that there are n tiles with an α angle at V , k tiles with a β angle at V , and j tiles with a γ angle at V . Since α is not a rational multiple of π , it follows from $3\alpha + 2\beta = \pi$ that α is not a rational multiple of β or γ , and $\gamma = \beta + 2\alpha$ is not a rational multiple of β . Each vertex is therefore of one of the types $(1, 1, 1)$, $(2, 2, 2)$, $(0, 1, 3)$, $(3, 2, 0)$, $(6, 4, 0)$, and $(4, 3, 1)$. Vertices of types $(1, 1, 1)$ and $(2, 2, 2)$ are called “standard vertices”. A vertex of type $(0, 1, 3)$ is called a “center”. Vertices of types $(3, 2, 0)$, $(6, 4, 0)$, and $(4, 3, 1)$ are called “sporadic vertices”, since they do not occur in any known tilings.

Lemma 5 *Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . Suppose triangle ABC is N -tilled by a tile with angles α and β , and there are 5 angles of tiles altogether at the vertices of ABC . Then the number of centers is $N_C = 1 + N_1 + 2N_2$, where N_1 is the number of vertices of type $(3, 2, 0)$ and N_2 is the number of vertices of type $(6, 4, 0)$.*

Proof. In an N -tiling there are N triangles. Each has one α , one β , and one γ angle, so the total number of each kind of angle is N . At the vertices of ABC we have three α angles and two β angles, making an excess of three α angles over γ angles at the vertices of ABC . Similarly, at the sporadic vertices we get an excess of $3N_1 + 6N_2$. At the centers, the number of γ angles exceeds the number of α angles by 3 per center, for a total of $3N_C$. Therefore the total excess of α and β angles over γ angles is

$$0 = 3 + 3N_1 + 6N_2 - 3N_C$$

Solving for N_C we have

$$N_C = 1 + N_1 + 2N_2$$

as claimed. That completes the proof of the lemma.

4 The tiling equation

In this section we will derive the “tiling equation” mentioned above. The proof uses colorings of tilings. We think of coloring each tile black or white, in such a way that tiles touching along a line segment have different colors. Technically we can represent the two colors as ± 1 and speak of the “sign” of a triangle.

Lemma 6 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC is N -tiled by a tile with angles α and β . Suppose α is not a rational multiple of π . Then it is possible to assign a color “black” or “white” to each of the N tiles in such a way that along each interior edge of the tiling, the triangles on opposite sides of the edge receive opposite colors, and the tile at angle B is black.*

Proof. Let T_k be any one of the N tiles, and let P be a point in T_k . Let σ be a path from vertex B to P that does not pass through any vertex of the tiling, and passes transversally through each edge it crosses. (Transversally means it is not tangent to the edge.) Then the sign we wish to assign to T_k is the number of edges crossed by σ . We claim this sign does not depend on which path σ is chosen, but only on the tile T_k . To show that it suffices to show that the number is invariant under homotopies of σ fixing the two endpoints. That in turn follows from the fact that an even number of (segments of) edges meet at each interior vertex. We will now prove that. Since α is not a rational multiple of π , also α is not a rational multiple of β (since if it were, then it would follow from $3\alpha + 2\beta = \pi$ that α is a rational multiple of π). Therefore there are only the following types of interior vertices: (1) vertices where two each of α , β , and γ angles meet; (2) vertices where there are three γ angles and one β angle; (3) vertices with six α angles and four β angles; vertices with four α angles, three β angles, and one γ angle; (4) non-strict vertices occurring on an edge of some tile (rather than at its vertex) and with one each of α , β , and γ ; (5) non-strict vertices occurring on an edge with three α and two β angles. Any other combination of angles at the vertex will give another linear relation between α and β besides $3\alpha + 2\beta = \pi$, which will imply that α is a rational multiple of π . In each of the five cases enumerated, there are an even number of segments of edges meeting at the vertex. That completes the proof of the lemma.

Lemma 7 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles 2α and β and is N -tiled by a tile with angles α and β . Suppose α is not a rational multiple of π . Let the tiles be assigned signs in accordance with the previous lemma. Then we have*

$$M(a + b + c) = X - Y + Z$$

where M is the number of positive tiles minus the number of negative tiles.

Proof. Consider a “maximal segment” PQ , i.e. a part of a straight line consisting of interior edges of the tiling, which cannot be extended to a longer such segment (and hence has its endpoints at strict vertices P and Q , where the angle sum of the angles at the vertex is 2π). Then all the tiles on one side of PQ are positive, and all the tiles on the other side are negative. The signed sum of the edges all these triangles share with PQ is zero. Hence, the total length of the positive interior edges equals the total length of the negative interior edges.

On the boundary of ABC , all the tiles sharing edges with sides $Z = AB$ and $X = BC$ are positive (since the tile at B is positive); and since there are two tiles sharing vertex A and two tiles sharing vertex C , all the tiles on side $Y = AC$ are negative. Let M be the number of positive tiles minus the number of negative tiles. Then the difference between the total length of the positive

edges and the total length of the negative edges is M times the perimeter $a + b + c$ of a tile. Since the positive and negative interior lengths are equal, we have

$$M(a + b + c) = X - Y + Z$$

That completes the proof of the lemma.

Lemma 8 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles 2α and β and is N -tiled by a tile with angles α and β . Suppose α is not a rational multiple of π . Let the tiles be assigned signs in accordance with Lemma 6. Let $s = 2\sin(\alpha/2)$ and let M be the number of positive tiles minus the number of negative tiles. Then $M > 0$, and s is rational, and we have*

$$s^2 = \frac{2M^2}{M^2 + N}$$

Proof. By the previous lemma we have $M(a + b + c) = X - Y + Z$. By Lemma 1, we have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} at \\ b \\ c \end{pmatrix}$$

Hence $M(a + b + c) = \lambda(at - b + c)$. Dividing by c and using $t = 2 - s^2$ we have

$$\begin{aligned} M(s + (1 - s^2) + 1) &= \lambda(s(2 - s^2) - (1 - s^2) + 1) \\ M(s + 2 - s^2) &= \lambda(2s - s^3 + s^2) \\ &= \lambda s(2 - s^2 + s) \end{aligned}$$

Dividing both sides by $2 - s^2 + s$ (which cannot be zero since $0 < s < 1$) we have

$$M = \lambda s$$

Since λ and s are both positive, this implies $M > 0$. By the area equation, $N = \lambda^2 t$. We have

$$\begin{aligned} M^2 &= \lambda^2 s^2 \\ &= s^2 \frac{N}{t} \\ &= N \frac{s^2}{2 - s^2} \\ \frac{N}{M^2} &= \frac{2 - s^2}{s^2} \\ &= \frac{2}{s^2} - 1 \end{aligned}$$

Solving for s^2 we have

$$s^2 = \frac{2M^2}{M^2 + N} \tag{1}$$

the equation stated in the lemma. Hence s^2 is rational. Hence s is either rational or has degree 2 over \mathbb{Q} ; but if s is not rational, then it has degree at least 3 over \mathbb{Q} by Lemma 3. Hence s is rational. That completes the proof of the lemma.

Lemma 9 *Let $3\alpha + 2\beta = \pi$ and assume triangle ABC has angles 2α and β and is N -tiled by a tile with angles α and β . Suppose α is not a rational multiple of π . Then N satisfies the equation*

$$M^2 + N = 2K^2$$

for some integer M with $M^2 < N$ and some K , and N is a square times a product of distinct primes of the form $8n \pm 1$. Moreover $s = 2\sin(\alpha/2) = M/K$.

Proof. If there is an N -tiling as in the lemma, then by the previous lemma, $M^2 + N$ is twice a rational square, hence twice an integer square; hence $M^2 + N = 2K^2$ is solvable for K with $K > 0$. Hence 2 is a square mod p for each p dividing N but not M . But 2 is square mod p just in case p is congruent to ± 1 mod 8. Hence, primes dividing N but not M are congruent to ± 1 mod 8. If p is a prime that divides both N and M , but divides N to a power p^j that does not divide M^2 , then similarly 2 is a square mod p .

Now to prove $M^2 < N$. Recall from (1) that

$$s^2 = \frac{2M^2}{M^2 + N}$$

But $s = 2\sin(\alpha/2)$. Both α and β are less than γ , so both α and β are less than $\pi/3$. Hence $\alpha/2 < \pi/6$, so $\sin(\alpha/2) < 1/2$, and $s < 1$. Hence $2M^2/(M^2 + N) < 1$. Hence $2M^2 < M^2 + N$; hence $M^2 < N$ as claimed in the statement of the lemma. Since $M^2 + N = 2K^2$ we have $s^2 = 2M^2/(M^2 + N) = M^2/K^2$; taking the square root of both sides we have $s = M/K$. That completes the proof of the lemma.

Remark. Thus any odd primes p dividing N that are not congruent to ± 1 mod 8, must occur to even powers and divide M^2 to at least the power they divide N .

Question: If I divide M and K both by 2 (assuming they are both even), then $s = M/K$ does not change, and since $s = 2\sin(\alpha/2)$ determines the shape of the tile, the new tile is similar to the old one, just half the linear dimensions, so one-quarter the area. But N is divided by 4 according to the tiling equation, whereas, it should take four times as many tiles to tile ABC with these smaller tiles. What is going on here? *Answer:* When you divide M and K by 2, the size of the tiled triangle ABC also changes. Since the length Y of side AC is λb , and $\lambda = K$, if we divide K and b both by 2 then side Y is divided by 4, so the area of ABC is 16 times smaller. The tiles are four times smaller, so indeed it takes only one-fourth as many of them to tile the new ABC , not four times as many.

5 For each N , there are only a few possibilities for the tile and for ABC

Namely, the equation $M^2 + N = 2K^2$ with $M < N$ can have at most finitely many solutions in integers M and K ; and if the solution comes from a tiling, then we do have $M < N$, since M is the number of positive tiles minus the number of negative tiles, so it is less than the total number of tiles.

There are comparatively few such M and K , often just one for a given N . Since $s = 2\sin(\alpha)/2 = M/K$, each solution determines one possible value of α , and then the eigenvalue λ and the number t are also determined, as follows:

$$\begin{aligned}
s &= \frac{M}{K} \\
\lambda &= \frac{M}{s} = K \\
t &= 2 - s^2 \\
&= s^2 \frac{N}{M^2} \\
&= \left(\frac{M}{K}\right)^2 \frac{N}{M^2} \\
&= \frac{N}{K^2} \\
1 - s^2 &= \frac{N - K^2}{K^2}
\end{aligned}$$

Writing the \mathbf{d} matrix equation in the form

$$\begin{aligned}
ps + d(1 - s^2) + e &= \lambda st \\
gs + m(1 - s^2) + f &= \lambda(1 - s^2) \\
hs + \ell(1 - s^2) + r &= \lambda
\end{aligned}$$

and substituting the expressions for t , λ and s found above, and multiplying by K^2 , we find

$$\begin{aligned}
pMK + d(N - K^2) + eK^2 &= MN \\
gMK + m(N - K^2) + fK^2 &= K(N - K^2) \\
hMK + \ell(N - K^2) + rK^2 &= K^3
\end{aligned}$$

The tile (a, b, c) is thus similar to the triangle with sides $(MK, N - K^2, K^2)$, and if we choose those values for (a, b, c) , instead of the proportional $(\sin \alpha, \sin \beta, \sin \gamma)$, we find that the sides of ABC are $(MN, K(N - K^2), K^3)$. When we choose a particular N , it may be convenient to cancel the greatest common divisor of this (a, b, c) . In particular if K divides N , then K can be canceled. This choice shows that a , b , and c can be taken to be integers; but we usually find it more convenient to take $a = M$, $c = K$, and $b = (N/K) - K = (N - K^2)/K = (K^2 - M^2)/K = K(1 - (M/K)^2) = c(1 - (M/K)^2)$. With this choice of a and c , b is an integer if and only if K divides M^2 .

We note that the equation does sometimes have more than one solution. The least such N is $N = 119$, where we have $(M, K) = (3, 8)$ or $(9, 10)$. These yield tiles $(357, 440, 512)$ and $(1071, 190, 1000)$. In that case K does not divide N , and as we will see below, no tiling exists in that case. The least N such that the equation has more than one solution in which K divides N is $N = 87808$. In that case there are two solutions with K dividing N , namely $M = 112, K = 224$ and $M = 208, K = 256$. These correspond to the tiles $(112, 119, 224)$ and $(208, 87, 256)$ respectively. The corresponding triangles ABC are quite large; it is not practical to make a picture that fits on one page.

6 No N -tilings for $N < 28$

Logically, this section is unnecessary, since the result is a special case of our main theorem below. Nevertheless it is of some interest that it can be proved without the concepts (e.g. connected components of a tiling) that we introduce to prove the main theorem. In fact we give two direct proofs for $N = 7$ here.

Lemma 10 *Let $3\alpha + 2\beta = \pi$ and suppose triangle ABC has angles 2α and β , and α is not a rational multiple of π . Then there is no N -tiling of ABC for $N < 28$.*

Proof. For $N < 28$, the equation $M^2 + N = 2K^2$ has solutions with $M < K$ (corresponding to $s < 1$) only for $N = 7, 14, 17$, and 23 . Since N must be a square times a product of primes congruent to $\pm 1 \pmod 8$, we can eliminate 14 and 17; only 7 and 23 need to be considered. Note that we have now narrowed the search so much that there is only one possible tile (for each N) and only one possible triangle ABC to consider.

First assume $N = 7$. Then the only solution of $M^2 + N = 2K^2$ is $M = 1$ and $K = 2$, so the equations above show that a triangle similar to the tile will have sides $MK = 2$, $N - K^2 = 3$, and $K^2 = 4$, and with this tile (2-3-4 for short), the triangle ABC will have sides $MN, K(N - K^2), K^3$, or 7-6-8.

There are very few possibilities for the \mathbf{d} -matrix. The first row of the \mathbf{d} -matrix must express 7 as an integral combination (with nonnegative coefficients) of 2, 3, and 4, which can only be done as $7 = 3 + 4$, so the first row can only be $(0, 1, 1)$. The second row must express AC , which is 6, as an integral combination of 2, 3, and 4, so it could be $(3, 0, 0)$ or $(0, 2, 0)$ or $(1, 0, 1)$. But since at vertex B there are two α angles, the triangles at vertex B have their a sides in the interior, so there must be at least one b or c side along BC . Hence $(3, 0, 0)$ can be ruled out. Suppose the second row is $(1, 0, 1)$. Then the tile T_1 at vertex B has its c side along BC . Since $c = 4$ and $BC = 6$, there is room for only one more tile T_2 along BC , and it must have its a side on BC , so it cannot have its α angle at C ; it must have its β angle there, since vertex C is composed of an α and a β angle. Then T_2 must have its c side on BC , and its γ angle at the vertex P on AC that it shares with T_1 . The third tile T_3 sharing vertex P therefore has its α angle at P , since T_1 and T_2 contribute β and γ respectively. Then T_3 has either its b or its c side shared (partly) with the a side of T_1 ; but then this side of T_3 extends beyond T_1 . Now there is not enough room for the tile T_4 that must share vertex A with tile T_1 . It must have its α angle at vertex A , so its a side cannot be shared with T_1 . Hence the side it shares with T_1 must be its b or c side, which is impossible because that would meet the interior of T_3 . This contradiction shows that the second row of the \mathbf{d} -matrix cannot be $(1, 0, 1)$. The only remaining possibility is $(0, 2, 0)$, i.e., side AC is composed of two b sides. Then tile T_1 has its α angle at vertex A , its b side along AC , and its γ angle at the center P of side AC . Tile T_2 has its b side equal to PC , and hence it does not have its β angle at C . Hence it has its α angle at C . Since its β angle is opposite PC , it must have its γ angle at P . But now there are two γ angles at P , which is impossible, since $\gamma > \pi/2$. That completes the proof that there is no 7-tiling.

Now assume $N = 14$. The only solution of $M^2 + 14 = 2K^2$ with $M^2 < 14$ is $M = 2$ and $K = 3$, so the shape of the tile is $(MK, N - K^2, K^2)$, which is 6-5-9. Triangle ABC has the shape $(MN, K(N - K^2), K^3)$, which is 28-15-27. Consider the second row of the \mathbf{d} -matrix, which expresses side $AC = 15$ as an integral combination of $a = 6$, $b = 5$, and $c = 9$. There are only two possibilities: $(0, 3, 0)$, corresponding to three b sides ($15 = 3 \cdot 5$), and $(1, 0, 1)$, corresponding to one a side and one c side ($15 = 6 + 9$). Suppose it is $(0, 3, 0)$. Tile T_1 at A has its α angle at A and its b side on AC , so its β angle is in the interior and its γ angle is at vertex V on AC . Let T_2 be the other tile on AC , with its b side on AC . Its γ angle cannot be at V because that would make 2γ , which is more than π . Its γ angle cannot be at C since the total angle at C is $\alpha + \beta < \gamma = 2\alpha + \beta$. But the γ angle of T_2 must be at V or at C , since the other vertex of T_2 is opposite side b and must therefore have the β angle. This contradiction shows that $(0, 3, 0)$ cannot be the second row of the \mathbf{d} -matrix.

Therefore it must be the other possibility, $(1, 0, 1)$. Let T_1 be the tile at vertex A sharing side AC . Since it has its α angle at A , it does not have its a side on AC . Hence its c side is on AC , and the other tile T_2 on AC has its a side on AC . Therefore it has its β angle at C , since there is not room at C for a γ angle. Now consider tile T_3 , which shares a vertex V on AC with T_1 and T_2 .

Since T_2 has its γ angle at V and T_1 has its β angle at V , T_3 must have its α angle there. Hence the side of T_3 that it partially shares with T_1 is not its a side. If it were the c side then T_3 would extend past T_1 and there would be no room to place a second tile at vertex A . Hence the side of T_3 along T_1 must be its b side, which is 5, while $a = 6$, so it does not quite reach to the interior vertex W of T_1 . Then no matter what tile is laid at vertex A next to T_1 , that tile will either extend past the short side $b = 5$ of T_1 , or just reach the end of that side at W and then extend past the line VW . In either case there will be a segment one unit long between W and tile T_2 that can never have a tile placed adjacent to it. That completes the proof for $N = 14$.

Another way to settle the case $N = 14$ is by computer. Computer search for a boundary tiling, which is discussed in the last section of this paper, finds no boundary tilings, and hence there are no 14-tilings. The program starts from vertex B , and sometimes makes it down side BA and onto side AC , but it never finds a partial tiling that goes past vertex C ; and the above proof provides a clear explanation why. $N = 14$ is a simple case: there are only 365 backtracking points, i.e. partial boundary tilings that are abandoned because they cannot be completed.

Now assume $N = 23$. The only solution of $M^2 + 23 = 2K^2$ with $M^2 < 23$ is $M = 3$ and $K = 4$, so the shape of the tile is $(MK, N - K^2, K^2)$, which is 12-7-16, and triangle ABC has the shape $(MN, K(N - K^2), K^3)$, which is 69-28-64. Note that here we have $a > b$, which is allowed since in this section we are not assuming $\alpha < \beta$. The second row of the \mathbf{d} -matrix has to express 28 as an integral combination of 12, 7, and 16. So the possibilities are $(1, 0, 1)$ and $(0, 4, 0)$. We can rule out $(0, 4, 0)$ since the tile T_1 at vertex A has to have its b or c side on AC . Hence the second row of the \mathbf{d} -matrix is $(1, 0, 1)$, and as in the case $N = 7$ there are just two tiles along AC . Tile T_1 thus has its c side on AC , and tile T_2 has its a side on AC ; those two tiles share vertex P , where T_1 has a β angle (since its γ is opposite AP and its α is at A), and tile T_2 has its a side on PC , and hence does not have its α angle at C ; so it must have its β angle at C and its α angle opposite PC , so it must have its γ angle at P . Now T_1 has a β angle at P and T_2 has a γ angle there, so the third tile T_3 sharing vertex P has an α angle there, and hence does not have its a side shared with T_1 . Consider the other tile T_4 at vertex A . Like T_1 , it has its α angle at A , so it has its b or c side shared with T_1 . If it shares the b side, then both T_1 and T_2 have their γ angles at a shared vertex Q ; if T_4 has its c side along T_1 , then it extends beyond T_1 . In either case then, it is not possible for tile T_3 to extend past T_1 along their shared boundary. Since tile T_1 has its a side there, tile T_3 either has its a or b side along T_1 . But we already proved it does not have its a side there. Hence it has its b side there. But that leaves a section of the boundary of T_1 that does not touch T_3 , of length $a - b$. That is $12 - 7 = 5$. Since 5 is less than the shortest side b of the tile, there is no way to place a tile along this segment. That completes the proof that there is no 23-tiling, and that in turn completes the proof of the lemma.

7 The case $N = 28$

The equation $M^2 + 28 = 2K^2$, with $M^2 < N$, has only one solution, namely $M = 2$ and $K = 4$. That yields the tile 2-3-4 and the triangle ABC is 14-12-16. This is the same shape tile and triangle as in the case $N = 7$, except now the triangle is twice as big in linear dimensions. We work out what its angles are as follows:

$$\begin{aligned} s^2 &= \frac{2M^2}{M^2 + N} \\ &= \frac{8}{32} \quad \text{since } M = 2 \text{ and } N = 28 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \\
s &= \frac{1}{2} \\
&= 2 \sin \frac{\alpha}{2} \quad \text{by definition of } s
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\alpha}{2} &= \arcsin \frac{s}{2} \\
&= \arcsin \frac{1}{4} \\
\alpha &= 2 \arcsin \frac{1}{4} \\
&= 28.9550244^\circ
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{b}{c} &= 1 - s^2 = \frac{3}{4} \\
c &= \sin \gamma \\
&= \cos \frac{\alpha}{2} \quad \text{by Lemma 2} \\
&= \sqrt{1 - \sin^2 \frac{\alpha}{2}} \\
&= \sqrt{1 - \frac{s^2}{4}} \\
&= \sqrt{1 - \frac{1}{16}} \\
&= \sqrt{\frac{15}{16}}
\end{aligned}$$

Hence

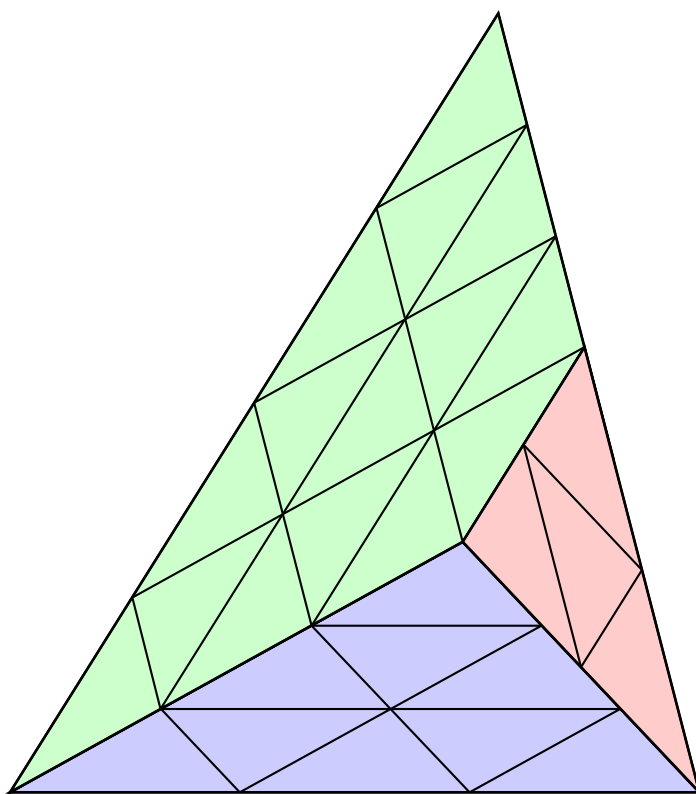
$$\begin{aligned}
\sin \beta &= b \\
&= \left(\frac{b}{c}\right)c \\
&= \frac{3}{4} \sqrt{\frac{15}{16}} \\
&= \frac{3\sqrt{15}}{16} \\
\beta &= \arcsin \frac{3\sqrt{15}}{16} \\
&= 46.5674634^\circ
\end{aligned}$$

Now your calculator can verify that $2\alpha + 3\beta = 180^\circ$. One would not have known *a priori* that there was anything special about these angles, but since we have $a/c = s = 1/2$ and $b/c = 1 - s^2 = 3/4$, the triangle with sides a , b , and c is similar to a triangle with sides 2, 3, and 4, which is a much

easier way of describing the tile than by its angles. Then the candidate rows of the \mathbf{d} matrix show that the sides X , Y , and Z of the triangle ABC are respectively 14, 12, and 16. Since $2\alpha > \beta$ in this case, side X , which is opposite the 2α angle, is greater than side Y , which is opposite the β angle, and the largest side, 16, is opposite the $\alpha + \beta$ angle.

For some time I thought that there was no 28-tiling. I could not find one by hand searching, even with the aid of paper tiles. I then wrote a computer program to search for one, fully expecting to show that none exist. But on October 7, 2011, the still-buggy program was producing “boundary tilings”, placing triangles in many possibly ways around the boundary of ABC , and I saw that one of these can be filled in. You can see the tiling in Fig. 1.

Figure 1: A 28-tiling



8 Triquadratic tilings

In this section we generalize the 28-tiling, showing that it is just the simplest member of a new family of tilings.

Lemma 11 *Suppose that in triangle T , whose sides are a , b , and c , we have $b = c - a^2/c$. Let α and β be the angles opposite a and b respectively. Then $3\alpha + 2\beta = \pi$.*

Proof. Since the condition $b = c - a^2/c$ is invariant if the triangle is re-scaled, we may as well assume $c = 1$. Then we have $b = 1 - a^2$, and after the rescaling we will have $a < 1$. We could not find a high-school-trigometry proof of this lemma; we had to use a little calculus. By the law of cosines we have the following two equations:

$$\begin{aligned} a^2 &= 1 + (1 - a^2)^2 - 2(1 - a^2) \cos \alpha \\ (1 - a^2)^2 &= 1 + a^2 - 2a \cos \beta \end{aligned}$$

Solving for α and β we have

$$\begin{aligned} \alpha &= \arccos\left(\frac{1 + (1 - a^2)^2 - a^2}{2(1 - a^2)}\right) = \arccos(1 - a^2/2) \\ \beta &= \arccos\left(\frac{1 + a^2 - (1 - a^2)^2}{2a}\right) = \arccos\left(\frac{3a - a^3}{2}\right) \end{aligned}$$

Now we form the expression $3\alpha + 2\beta$ and differentiate it with respect to a . When we differentiate \arccos we get an algebraic function. After the differentiation we simplify and show that we get zero. That will prove that $3\alpha + 2\beta$ is a constant. Then we evaluate the constant using one particular triangle and find that it is zero. Here are the details:

$$\begin{aligned} 3\alpha + 2\beta &= 3 \arccos(1 - a^2/2) + 2 \arccos\left(\frac{3a - a^3}{2}\right) \\ \frac{d}{da}(3\alpha + 2\beta) &= 3 \frac{a}{\sqrt{1 - (1 - a^2/2)^2}} - 2 \frac{(3 - 3a^2)/2}{\sqrt{1 - (3a - a^3)^2/4}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\sqrt{1 - (3a - a^3)^2/4}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\sqrt{(1 - (3a - a^3)/2)(1 + (3a - a^3)/2)}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{3 - 3a^2}{\frac{1}{2}\sqrt{(a^3 - 3a + 2)(a^3 + 3a + 3)}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{6(1 - a^2)}{\sqrt{(a + 2)(1 - a)^2(a - 2)(a + 1)^2}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{6(1 - a^2)}{(1 - a)(1 + a)\sqrt{(a + 2)(a - 2)}} \\ &= \frac{6}{\sqrt{4 - a^2}} - \frac{6}{\sqrt{4 - a^2}} \\ &= 0 \end{aligned}$$

as promised. Hence $3\alpha + 2\beta$ is a constant. Now, to show that the constant in question is zero, it suffices to evaluate $3\alpha + 2\beta = 0$ for a particular value of a . Or, approaching the matter another way, let $\alpha = 30^\circ$ and $\beta = 45^\circ$, so $3\alpha + 2\beta = \pi$, and let us check that the triangle with those angles and long side 1 has sides a and $1 - a^2$ for some number a . Let a and b be the sides of that triangle opposite the 30° and 45° angle respectively. We must show $b = 1 - a^2$. We have

$$\begin{aligned} \sin 105^\circ &= \sin 75^\circ \\ &= \sin(30^\circ + 45^\circ) \end{aligned}$$

$$\begin{aligned}
&= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\
&= \frac{1}{2} \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \frac{1}{\sqrt{2}} \\
&= \frac{1 + \sqrt{3}}{2\sqrt{2}}
\end{aligned}$$

By the law of sines we have

$$\frac{b}{\sin 45^\circ} = \frac{a}{\sin 30^\circ} = \frac{1}{\sin 105^\circ}$$

Putting in the values of the trig functions we have

$$\begin{aligned}
\frac{b}{1/\sqrt{2}} &= \frac{a}{1/2} = \frac{2\sqrt{2}}{1 + \sqrt{3}} \\
b\sqrt{2} &= 2a = \frac{2\sqrt{2}}{1 + \sqrt{3}} \\
a &= \frac{2}{1 + \sqrt{3}} \\
b &= \frac{2}{1 + \sqrt{3}} \\
1 - a^2 &= \frac{(1 + \sqrt{3})^2 - 2}{(1 + \sqrt{3})^2} \\
&= \frac{2}{1 + \sqrt{3}} \\
&= b
\end{aligned}$$

as claimed. That completes the proof of the lemma.

Theorem 1 *Let N be given. Suppose $M^2 + N = 2K^2$ with $M^2 < N$ and suppose K divides N (or equivalently, K divides M^2). Let ABC be the triangle with sides $BC = MN/K$, $AC = N - K^2$, and $AB = K^2$, and let the triangle T have sides $a = M$, $b = N/K - K$ (which must be a positive integer), and $c = K$. Then there is an N -tiling of triangle ABC by tile T . This tile satisfies $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π .*

Proof. First note that since $M^2 < N$, we have $2K^2 = M^2 + N < 2N$, so $K^2 < N$. Hence $K < N/K$. Hence b , which is defined to be $N/K - K$, is positive. Since K divides N , b is a positive integer.

Define $J = c - b = K - (N/K - K) = 2K - N/K = (2K^2 - N)/K = M^2/K$. Since K divides N , J is an integer. We then have $c = K = M^2/J$ since $J = M^2/K$. Since $a = M$ and $b = c - J$ we have

$$\begin{aligned}
a &= M \\
c &= \frac{a^2}{J} \\
b &= \frac{a^2}{J} - J
\end{aligned}$$

Let α , β , and γ be the angles of the tile opposite sides a , b , and c . Then we have $b = c - a^2/c$ since $b = c - J$ and $J = M^2/K = a^2/c$. By Lemma 11 we have $3\alpha + 2\beta = \pi$.

We can construct a tiling as follows. Fix the “center point” Q . Construct three quadratic tilings whose vertex angles meet at Q , one with a^2 tiles, and two with b^2 tiles. Let the two b^2 tilings share a common side AQ . Let the a^2 tiling share a common side CQ with one of the b^2 tilings. This is possible because along CQ there are, on the side with the a^2 tiling, a tiles, each with its b edge on that side, so CQ has length ab ; on the side of CQ with the b^2 tiling, there are b tiles, but each tile has its a edge along QC , so the length on that side is also ab . Hence the corners of the a^2 tiling and the b^2 tiling occur at the same point C . There are two α angles at A and at C there are an α and a β angle. Let D be the other vertex of the b^2 tiling that does not share side QC . Let E be the other vertex of the a^2 tiling, where the corner tile has a β angle. Now construct point B as the intersection of the line containing AD and the line containing CE . Since angle DAC is 2α and angle ACE is $\alpha + \beta$, the sum of these two angles is $3\alpha + \beta < \pi$. Hence, by Euclid’s fifth postulate, point B does exist and lies on the same side of AC as the tilings.

Now consider the quadrilateral $BDQE$. The interior angle at B is β , since the angles of triangle ABC must sum to π , and the angles at A and C sum to $3\alpha + \beta = \pi - \beta$. The exterior angle QEC is also β , so AB is parallel to QE . The exterior angle ADQ is also β , so BC is parallel to DQ . Hence quadrilateral $BDQE$ is a parallelogram. Its side QD is composed of b edges of tiles, each of length a . We can therefore divide quadrilateral $BDQE$ into b parallelograms by drawing $b - 1$ equally spaced lines parallel to EB . Each of these parallelograms has a side equal to QE . QE is composed of a edges of tiles, with each edge of length a , so the length of QE is a^2 . But $a^2 = Jc$. It is therefore possible to break each of the b small parallelograms into J yet smaller parallelograms, with sides c and a , and one angle γ . Each of these smaller parallelograms can be cut into two copies of the tile T . We can thus tile quadrilateral $BDQE$ by $2bJ$ tiles. (Note that these tiles extend the b^2 quadratic tiling of QAD , so they would be part of a c^2 quadratic tiling, but the vertex of that larger quadratic tiling would overlap the a^2 tiling.)

It remains to count the tiles and verify that there are N of them. The total number of tiles, which we temporarily call Z until we prove it is equal to N , is

$$Z = 2b^2 + a^2 + 2bJ$$

We substitute $a = M$ and $b = N/K - K$, obtaining

$$Z = 2(N/K - K)^2 + M^2 + 2(N/K - K)J$$

Now substitute $N = 2K^2 - M^2$; then $N/K - K = 2K - M^2/K - K = K - J$, since $J = M^2/K$. We obtain

$$\begin{aligned} Z &= 2(K - J)^2 + M^2 + 2(K - J)J \\ &= 2K^2 - 4KJ + 2J^2 + M^2 + 2KJ - 2J^2 \\ &= 2K^2 - 2KJ + M^2 \\ &= 2K^2 - 2K(M^2/K) + M^2 \quad \text{since } J = M^2/K \\ &= 2K^2 - 2M^2 + M^2 \\ &= 2K^2 - M^2 \\ &= N \end{aligned}$$

as desired. The number of tiles is N . That completes the proof of the theorem.

Figures 2, 3, and 4 illustrate triquadratic tilings. Note that in Fig. 3, we have $\alpha > \beta$ since $M > K/2$. (One can show using $s = 2\sin(\alpha/2) = M/K$, that $M > K/2$ is equivalent to $\alpha > \beta$.)

Figure 2: A triquadratic tiling with $N = 153 = 9 \cdot 17$, $M = 3$, $K = 9$

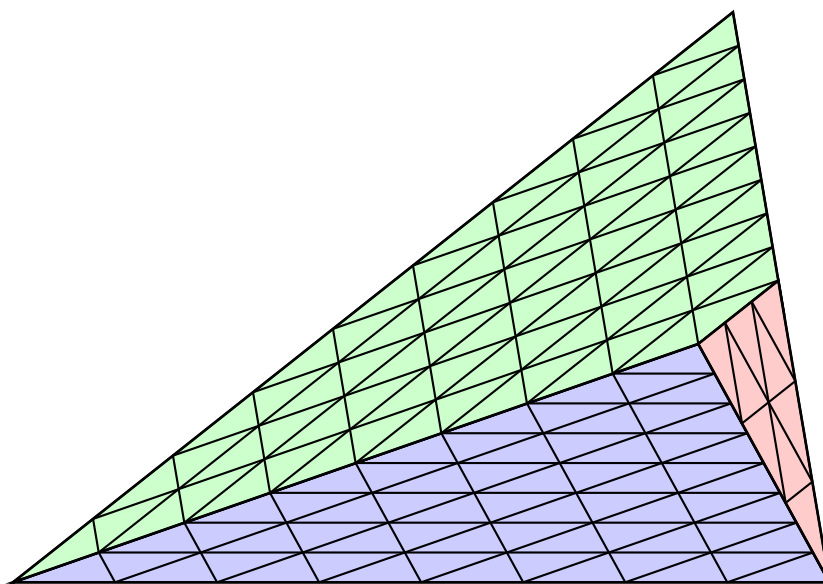


Figure 3: A triquadratic tiling with $N = 126 = 9 \cdot 14$, $M = 6$, $K = 9$.

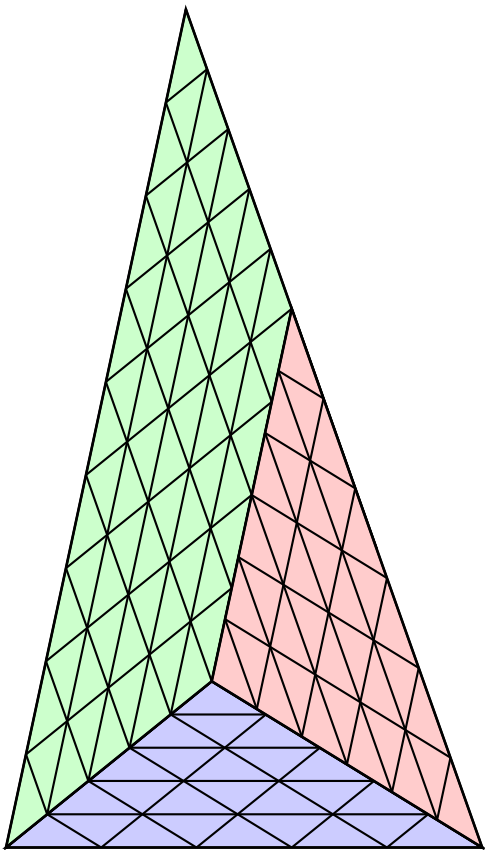
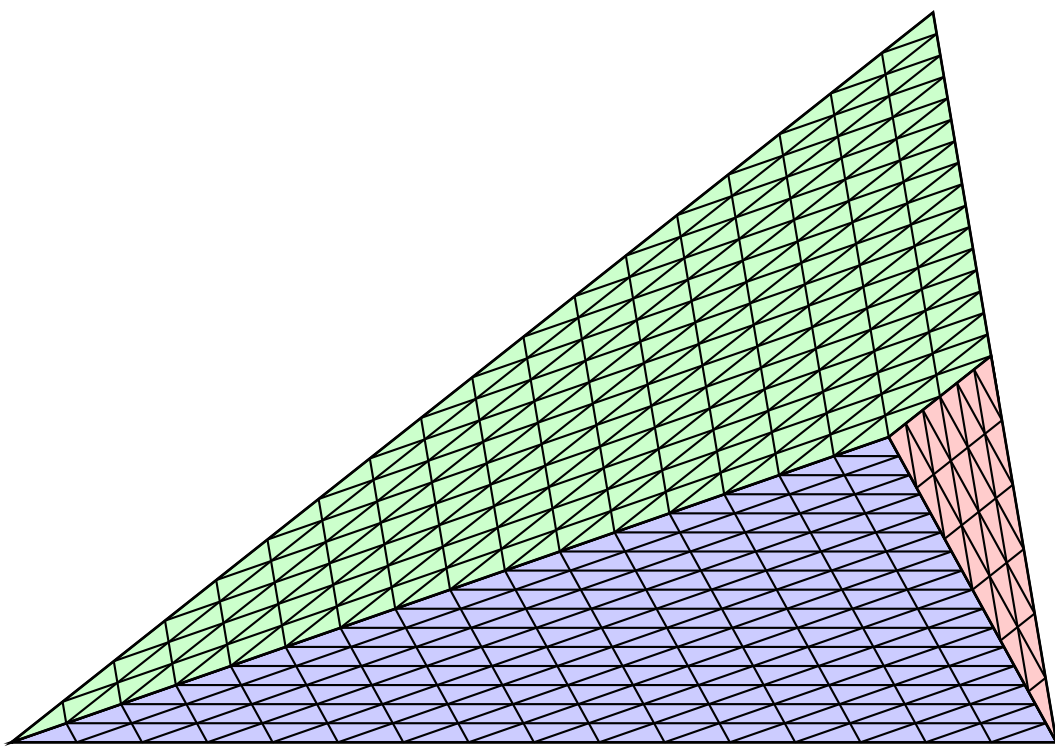


Figure 4: A triquadratic tiling with $N = 612 = 17 \cdot 6^2$, $M = 6$, $K = 18$



9 The \mathbf{d} -matrix reconsidered

Recall that the \mathbf{d} matrix tells us how many a , b , and c edges compose each side of ABC . In our other papers on triangle tiling, we considered the subfields of $\mathbb{Q}(a, b, c)$, but in this paper, they all collapse to \mathbb{Q} . Nevertheless, there is still information to be extracted from the relations of a , b , and c , since if K does not divide N , then (as it turns out), b is not an integer when $a = M$ and $c = K$, so there are number-theoretical limitations on how many c sides can equal how many b sides, etc. We begin by reviewing terminology and repeating a couple of short lemmas from previous work.

Generally our convention is that the sides of ABC are X , Y , and Z are in order of size, so X is opposite the smallest angle A , and Y is opposite the middle angle B , and Z is opposite the largest angle C . In this paper, however, we are not assuming $\alpha < \beta$, and that means that we may not have $Y < Z$ either. In case triangle ABC has angles α , 2α , and 2β , we will assume that X is opposite the α angle at A , and the 2α angle is at B , opposite Y , and that Z is opposite the 2β angle. In case ABC has angles 2α , β , and $\beta + \alpha$, we will assume X is opposite the 2α angle at A , and Y is opposite the β angle at B , and Z is opposite the $\beta + \alpha$ angle at C . Thus in case $\beta < \alpha$, we may not have X, Y, Z in order of size.

Lemma 12 *Suppose triangle ABC is N -tiled by a tile in which $\gamma > \pi/2$ (as is the case when $3\alpha + 2\beta = \pi$). Suppose all the tiles along one side of ABC do not have their c sides along that side of ABC . Then there is a tile with a γ angle at one of the endpoints of that side of ABC .*

Proof. Let PQ be the side of ABC with no c sides of tiles along it. Then the γ angle of each of those tiles occurs at a vertex on PQ , since the angle opposite the side of the tile on PQ must be α or β . Let n be the number of tiles along PQ ; then there are $n - 1$ vertices of these tiles on the interior of PQ . Since $\gamma > \pi/2$, no one vertex has more than one γ angle. By the pigeonhole principle, there is at least one tile whose γ angle is not at one of those $n - 1$ interior vertices; that angle must be at P or Q . That completes the proof of the lemma.

We use (as in previous papers) the following letters for elements of the \mathbf{d} -matrix:

$$\mathbf{d} = \begin{pmatrix} p & d & e \\ g & m & f \\ h & \ell & r \end{pmatrix}$$

Lemma 13 *Suppose $3\alpha + 2\beta = \pi$, and suppose triangle ABC has angles 2α , β , and $\beta + \alpha$. Let T be a triangle with angles α , β , and γ and suppose that ABC is N -tiled by T in such a way that just five tiles share vertices of ABC . Then the following restrictions on the elements of the \mathbf{d} matrix apply: We have $e \neq 0$, $f \neq 0$, and $r \neq 0$. In other words, the third element in each row is nonzero: there is a c edge on each side of ABC .*

Proof. None of the angles of ABC is large enough to accommodate the γ vertex of a tile, since those angles are 2α , β , and $\beta + \alpha < \beta + 2\alpha = \gamma$. The lemma then follows from Lemma 12, since the three numbers e , f , and r are the numbers of c sides of tiles on the three sides of ABC . That completes the proof of the lemma.

Suppose there is an N -tiling and $M^2 + N = 2K^2$; then we have $s = 2 \tan(\alpha/2) = M/K$, and $a/c = s$, $b/c = 1 - s^2$, and $t = 2 - s^2$, so we can take $a = M$, $c = K$, and $b = (N/K) - K$. Then $t = 2 - s^2 = (2K^2 - M^2)/K^2 = N/K^2$. We assume that K does not divide N , so b is not an integer. The denominator of N/K is K , if N and K are relatively prime, but otherwise it might be smaller.

We have $\lambda = K$, and the three sides of ABC are given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} at \\ b \\ c \end{pmatrix} = \mathbf{d} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Writing out the three rows of this matrix equation we have (by Lemma 1)

$$\begin{aligned} \lambda at &= pa + db + ec \\ \lambda b &= ga + mb + fc \\ \lambda c &= ha + \ell b + rc \end{aligned}$$

Lemma 14 *If $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π , and there is an N -tiling, and $N + M^2 = 2K^2$, then $X = MN/K$, $Y = N - K^2$, $Z = K^2$, and the \mathbf{d} -matrix equation takes the form*

$$\begin{aligned} MN/K &= pM + d(N/K - K) + eK \\ N - K^2 &= gM + m(N/K - K) + fK \\ K^2 &= hM + \ell(N/K - K) + rK \end{aligned}$$

Proof. Starting with the form of the equation(s) given just above the lemma, we note that the second equation implies $m \leq \lambda$, since the first and third terms on the right are nonnegative; and in fact we have $f > 0$ by Lemma 13; so $m < \lambda = K$. Putting in $\lambda = K$, $c = K$, $a = M$, $b = N/K - K$, and $t = N/K^2$, we have the stated equations. That completes the proof of the lemma.

Lemma 15 *If $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π , and there is an N -tiling, and $N + M^2 = 2K^2$, then $X = BC = M(b + c)$.*

Proof. The length of side BC (opposite vertex A) is λat according to the \mathbf{d} matrix equation, and the first row of the \mathbf{d} matrix (given in Lemma 14) tells us that it is MN/K . We have $a = M$, $c = K$, $b = c(1 - s^2) = K(1 - (M/K)^2) = (K^2 - M^2)/K = (N - K^2)/K = N/K - K$. Hence $b + c = (N/K - K) + K = N/K$. Hence $BC = M(b + c)$. That completes the proof of the lemma.

The same result can also be obtained directly from the relation $X - Y + Z = M(a + b + c)$, since $Y = \lambda b = Kb$ and $Z = \lambda c = Kc$ (by Lemma 1). So $X = BC = M(a + b + c) - Kc + Kb$; now putting $a = M$ and $c = K$ we have $X = M(b + c) + M^2 - K^2 + Kb$, and since $KB = K^2 - M^2$ the last three terms cancel, leaving $X = M(b + c)$.

10 No boundary tilings when $M = 1$

In the following theorem, a “boundary tiling” refers to a partial tiling containing enough tiles to cover the boundary of ABC and consistent with the requirement that there be at most one α , at most one β , and at most one γ at each boundary vertex.

Theorem 2 *Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . If $N + 1 = 2K^2$, i.e. if $M = 1$, then there is no N -tiling. In fact, there is not even a boundary tiling of ABC by the tile that would have to be used for an N -tiling.*

Proof. By Lemma 15, $BC = M(b + c)$. Since we now assume $M = 1$, we have $BC = b + c$. If $M = 1$ then K and N are relatively prime since K divides $N + 1$, so $b = N/K - K$ is not an integer and neither is $BC = b + c = N/K$.

Let T_1 be the tile at B . Let T_2 be the tile at C with an edge on BC . Since the other tile at C has its β angle at C , as shown in the previous lemma, tile T_2 has its α angle at C , because the total angle at C is $\alpha + \beta$. Hence T_2 has either its b or c side on BC .

Since T_1 has its β angle at B , the side that it shares with BC must be either its a side or its c side. We argue by cases on this alternative. Case 1, T_1 has its c side on AC . Let T_2 be the tile at C with an edge on BC . As shown above, T_2 has either its b or c side on BC , but it cannot be the c side, since together with the c side of T_1 , that would make $2c > b + c = AC$. Hence T_2 has its b side on AC . Hence its β angle is in the interior of ABC , opposite its b side. Hence T_2 has its γ angle at the vertex V that it shares with T_1 on BC . On the other hand T_1 has its β angle at B , and its γ angle on AB opposite its c side on BC , so it has its α angle at V . By Lemma 5, just one more tile (say T_3) shares vertex V , and T_3 has its β angle at b . Then the b side of T_3 is opposite vertex V , and hence is not on the line shared by T_1 and T_3 . However, the b side of T_1 , which has one end on BC at V and the other end on AC , is at least partly shared by T_3 . Since that is not the b side of T_3 , it must be the a side of T_3 , since the c side would extend outside of ABC across AC if placed alongside T_1 . But that leaves $b - a$ of that side of T_1 to be bordered by other tiles, which is impossible, because $a = M = 1$ is an integer, but b is not an integer, since K does not divide N . This contradiction shows that Case 1 cannot hold.

Case 2, T_1 has its a side on BC . As shown above, T_2 has its α angle at C and either its b or c side on BC . It cannot have its c side on BC , since we have one c edge and one a edge on side BC , and $BC = b + c$, so that would leave $(b + c) - c - a = b - a$ of BC yet to tile; but b is not an integer and so $b - a$ cannot be made up from a sides of tiles. Hence T_2 has its b side on BC . That leaves $c - a$ to be made up of edges of tiles other than T_1 and T_2 . No b tiles can be used, since $c - a$ is an integer, namely $K - 1$, and if we use one b edge we must use at least K of them to make an integer (since the denominator of b in lowest terms is K , but we cannot use K of them as $bK > c - a$. Here is the proof that $bK > c - a$:

$$\begin{aligned}
bK &= (N/K - K)K \\
&= N - K^2 \\
&= (2K^2 - M^2) - K^2 \\
&= K^2 - 1 \quad \text{since } M = 1 \\
&> K \quad \text{since } K \geq 2
\end{aligned}$$

Hence the b side of T_2 is the only b side of a tile on BC ; the rest of the tiles on BC have their a edges on BC . Since T_1 has its a edge on BC , its α angle is not on BC ; it has its β angle at B , so T_1 has its γ angle at its other vertex V on BC . other tiles with edges on BC ; since they have their a sides on BC , they do not have their α angles on BC ; so each of them has a β angle and a γ angle on BC . At each vertex on BC , there is exactly one β angle and one γ angle, by Lemma 5. Proceeding from B towards C , we see that each of these triangles is oriented the same way, with the β angle nearer to B than the γ angle. The last of these triangles, say T_j , shares a vertex W with tile T_2 . T_2 has its γ angle at W because its α angle is at C and its b side is on BC , so its β angle is not on BC . But T_j also has its γ angle at W , as shown above. Hence there are two γ angles at vertex W , contradicting Lemma 5. That completes the proof of the lemma.

Corollary 1 *There is no N -tiling for $N = 31, 71, 97, 127, 161$, or 199 ; in fact, not even a boundary tiling.*

Remark. These numbers are all primes or products of primes of the form $8n \pm 1$, so up until this theorem we could not rule out the existence of an N -tiling.

Proof. $31 + 1^2 = 2 \cdot 4^2$; $71 + 1^2 = 2 \cdot 6^2$; $97 + 1 = 2 \cdot 7^2$; $127 + 1^2 = 2 \cdot 8^2$; $161 + 1 = 2 \cdot 9^2$; $199 = 1^1 + 2 \cdot 10^2$.

Before proving the preceding theorem, we used a computer program to search for a boundary tiling. The output of this program was a pdf file containing pictures of rejected partial boundary tilings, rejected because they could not be completed. In Fig. 17, we display one of the 30,836 pictures the program produced before completing its search. The above proof is shorter and more comprehensible than the program, let alone its output.

11 The connected components of a tiling

We call a vertex in a tiling “of type 2π ” if the sum of the angles at that vertex is 2π , and “of type π ” if the sum of the angles is π . The latter occurs when the vertex is on the boundary of ABC , but it can also occur in the interior of ABC . In a tiling it may happen there exist line segments PQ bounding several tiles on one or both sides, such that the vertices of tiles on one side are not necessarily vertices of tiles on the other side. For example, one side of PQ might have three edges of length b and the other side might have two edges of length c . Such a segment is called an “interior segment” of the tiling if its endpoints are vertices of tiles on both sides of the edge (the definition is given more precisely below). The utility of this concept is that the sum of the lengths of the edges on one side must equal the sum of the length of the edges on the other side.

Definition 1 *An interior segment is a line segment formed of boundaries of tiles in a tiling, such that its endpoints are either vertices of type 2π , or cannot be extended past the endpoint and still lie on boundaries of tiles. An maximal segment is an interior segment that does not lie on any longer interior segment, i.e. cannot be extended (in either direction). An interior segment PQ that has the same number of edges of length a on each side of PQ , and the same number of edges of length b on each side, and the same number of edges of length c on each side, is called **inessential**. An interior segment that is not inessential is **essential**.*

Remarks. Equating the sum of the edges on one side of a interior segment to the sum of the edges on the other side gives rise to an integral relation (linear relation with integral coefficients) between a , b , and c . That relation will be trivial for an inessential interior segment, and nontrivial for an essential interior segment.

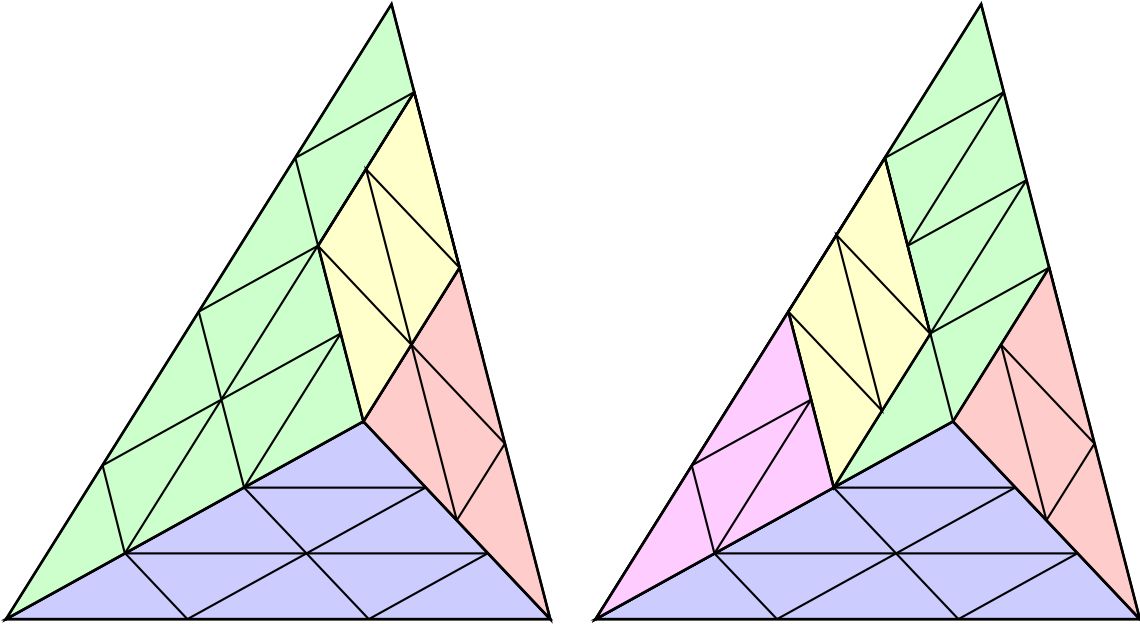
Examples. The 28-tiling illustrated in Fig. 1 contains two essential maximal segments. The one at the lower right in the figure has three a edges on one side and two b edges on the other. Thus for this tiling we have $3a = 2b$. which you can see near the lower right of the figure. Each biquadratic tiling contains a maximal segment on the altitude of ABC connecting the right angle to the hypotenuse.

Suppose given an N -tiling of triangle ABC , by the tile with angles α , β , and γ , where $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . We define a graph \mathbb{H} whose nodes are the tiles of the tiling. Since the word “edges” is already in use for the sides of a tile, we shall refer to the edges of this graph as “connections” instead, and speak of one tile being connected to another.

Definition 2 *Given a tiling of ABC , the graph \mathbb{H} has for its nodes the tiles used in the tiling, and its edges (connections) are defined as follows: Tiles T and S are connected if T and S share two vertices (and hence the edge between those vertices), and S and T do not have the same angles at the vertices of their shared side.*

Examples. Any quadratic tiling has only one connected component. Any biquadratic tiling has two connected components, one for each of the two quadratic tilings that it contains. Fig. 5 illustrates the connected components of two different 28-tilings, both different from the one in Fig. 1.

Figure 5: Components of two 28-tilings

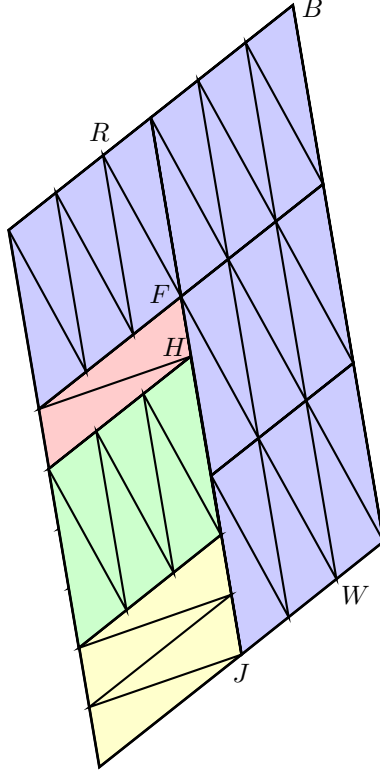


Remarks. We could have also allowed two tiles to be connected if they lie on opposite sides of an interior segment L and have edges of the same length lying on L , i.e. both have their a edges on L , or both have their b edges on L , or both have their c edges on L . This less stringent definition results in a graph with more connected components. For example, consider the partial tiling illustrated in Fig. 6. There are four components of the graph \mathbb{H} ; but with the less restrictive definition of connection, there would be only three. The segment FJ is a component boundary in \mathbb{H} , but with the less restrictive definition of connection, the tiles with c edges on FJ would be connected. It will turn out to be important that they *not* be connected.

Another variation on the definition of \mathbb{H} is the graph \mathbb{G} , defined by removing the requirement that connected tiles have different angles at their common vertices. In this graph, lines that have all the same length edges on both sides, with matching vertices but the tiles have equal angles at each vertex, are not component boundaries as they are in \mathbb{H} . For example, in the 28-tiling shown in in Fig. 1, such a line occurs as the angle bisector of angle A , and the two components of \mathbb{H} above and below that line will join into one \mathbb{G} -component. We will make no use of these variations on the definition of \mathbb{H} , and the word “component” will simply refer to component from now on.

Lemma 16 *Assume $a \neq b$, i.e. the tile is not isosceles. In any connected set C in a tiling, any two edges of the same length are parallel or on the same line. Two tile edges of different lengths lying on the same line do not belong to the same connected component,*

Figure 6: There are four connected components in this partial tiling, not three.



Remark. Of course a and b are always unequal if $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π , as we generally assume in this paper; but the lemma does not require those assumptions.

Proof. The second sentence of the lemma follows from the first. We prove the first by induction on the number n of tiles in \mathcal{C} . In the base case, when $n = 1$, if we have two edges of the same length they are the same edge, so they lie on the same line. That completes the base case. Now for the induction step. Suppose \mathcal{C} has $n + 1$ tiles. Let T be one of those tiles, and let S be a tile in \mathcal{C} that is directly connected to T ; then by the definition of the graph \mathbb{H} , the two tiles S and T form a parallelogram, whose opposite sides are parallel. Since $\alpha \neq \beta$, the sides of S and T of the same lengths are parallel (or on the same line, in the case of the shared side). We can apply the induction hypothesis to the set obtained by deleting T from \mathcal{C} ; then all the other tiles in \mathcal{C} have their corresponding edges parallel to or collinear with the edges of S , and hence to those of T . That completes the proof.

Definition 3 A **lattice tiling** (of some region, not necessarily a triangle) is a tiling whose vertices lie on the lattice points of some lattice.

For example, a quadratic tiling, or any subtiling of a quadratic tiling, is a lattice tiling.

Lemma 17 Suppose there is an N -tiling of ABC by the tile with angles α , β , and γ , and $\alpha \neq \beta$. Let \mathcal{E} be a connected component of the tiling, or more generally, the union of any connected set of tiles. Then the tiling (restricted to \mathcal{E}) is a lattice tiling of \mathcal{E} .

Remark. The 28-tiling shows that the lemma fails if the definition of \mathbb{H} is changed to permit tiles with the same angles at their shared vertices to be connected.

Proof. We proceed by induction on the number of tiles in the connected set \mathcal{E} . When this number is 1, the result is trivially true. Suppose \mathcal{E} is a connected set of $n + 1 \geq 2$ tiles. Let T be a tile on the boundary of \mathcal{E} . Remove T from \mathcal{E} , and call what is left \mathcal{E}' . By the induction hypothesis, the conclusion of the lemma holds for \mathcal{E}' . Let S be a tile in \mathcal{E} that shares an edge and its vertices with T . Then by the induction hypothesis, the vertices of S lie in a lattice that contains all the vertices of \mathcal{E}' . Since S and T form a parallelogram, that lattice also contains the third vertex of T . That completes the proof of the lemma.

The “type” of a component is specified by giving the directions of its a , b , and c sides. When we say a line segment has “direction AB ” we mean that it is parallel to AB , and similarly with AC or BC or other specified directions. In addition to the directions given by the sides of ABC , we will also need two other directions. “Direction C ” is given by the line separating the two tiles at C ; that line makes an angle of α with AC and an angle of β with BC , since every tile on BC has its β angle to the north. “Direction A ” is given by the angle bisector of angle BAC ; that bisector of course separates the two tiles with their α angle at A .

We give names to the following “types” of components:

Type I: c edges have direction BC , a edges have direction AB , and b edges have Direction C .

Type II: c edges have direction AB , a edges have direction BC , and b edges have Direction A .

Type III: c edges have direction AC , a edges have Direction A , and b edges have Direction C .

These three types of components are all illustrated in the 28-tiling and the other triquadratic tilings. But further types might also arise. For example, if there should be a component boundary along which tiles shared their a edges and vertices, but the tiles are oppositely oriented, and one of the components is Type I, then the other component will not be of any of the just-enumerated types. The concept of the type of a component, and the particular types Type I and Type II, will play an important role below. The reader is recommended to look at the pictures of the triquadratic tilings and identify the types of the components of those tilings. There are in principle infinitely many types, but they do not actually arise in tilings, so there is not much use in classifying them.

Technically, components are of Type I or Type II, not tiles. But we say that a tile is “of Type I” if it belongs to a component of Type I, and similarly for Type II.

Please have another look at Fig. 6. Note that there are two different Type I components in that figure, that are not connected even though some of their tiles share a common boundary with tile edges of the same length on that boundary. They still do not connect, since the tiles on that boundary do not share their vertices. This motivates the following definition:

Definition 4 *Two components \mathcal{C} and \mathcal{D} of the same type are said to be **out of sync** if they share a common boundary segment with edges of the same length on that boundary segment that do not share vertices.*

12 Bounds on the length of interior segments

Note that the greatest common divisor (K, N) of K and N satisfies $(K, N) = (K, M^2)$, since $N = 2K^2 - M^2$. Hence N and K are relatively prime if and only if M^2 and K are relatively prime; if and only if M and K are relatively prime. If $N + M^2 = 2K^2$, we have seen above that there exists an N -tiling if K divides M^2 .

Recall that $s = 2 \sin(\alpha/2) = M/K$, so we can take $a = M$ and $c = K$ and $b = N/K - K$. The geometric import of the condition that K does not divide M^2 is that when we take $a = M$ and $c = K$, b is an integer if and only if K divides M^2 . To prove this: $b = N/K - K = (2K^2 - M^2)/K - K = (K^2 - M^2)/K$, which is an integer if and only if K divides M^2 .

Lemma 18 *Suppose $jb = ua + vc$, where u, v , and j are integers. Suppose K does not divide M^2 . Then K divides j .*

Proof. Suppose $jb = ua + vc$. Let d be the greatest common divisor of M and K . Since $a = M$ and $c = K$, d divides $ua + vc = uM + vK$. Hence d divides jb , i.e. jb/d is an integer. Hence j/d is a multiple of the denominator of b . But $b = K - M^2/K$, so in lowest terms its denominator is K/d . That is, j/d is an integer multiple of K/d . Hence j is an integer multiple of K . That completes the proof of the lemma.

Lemma 19 *If $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π , and there is an N -tiling, and $N + M^2 = 2K^2$, then K divides mN . If K^2 does not divide M , then $m = \ell = 0$ and $d = M$ i.e. there are no b edges on AC and no b edges on AB , and $d = M$, i.e., there are exactly M edges of length b on BC .*

Proof. The second row of the \mathbf{d} -matrix equation is, according to Lemma 1,

$$Y = \lambda b = ga + mb + fc.$$

Since $\lambda = K$ we have $Kb = ga + mb + fc$. All the coefficients are nonnegative, so $m \leq K$, and subtracting mb we have $(K - m)b = ga + fc$. We have $m < K$ since if $m = K$ then $g = f = 0$, contradicting Lemma 13. Then $K - m > 0$, and then by Lemma 18, we have $K - m \geq K$, which implies $m \leq 0$; but since $m \geq 0$ that implies $m = 0$. That is, there are no b edges on AC .

By Lemma 15, the length of BC is $X = M(b + c)$. We have $X = pa + db + ec$. Therefore $X - Mb = Mc = pa + (d - M)b + ec$, or $(d - M)b = pa + (M - e)c$. Therefore by Lemma 18, we have $(d - M)$ congruent to zero mod K . We also have $X = \lambda b = Kb$, so $d \leq K$, so $d - M < K$. Since $M < K$ and $d > 0$, if $d < M$ we have $d > -K$, so $(d - M)$ cannot be congruent to 0 mod K in that case; hence $d \geq M$. Assume, for proof by contradiction, that $d \neq M$. Then we have $d - M \geq K$. Hence $d \geq K + M$. But $X = \lambda b = Kb$, so there cannot be more than K edges of length b on BC , i.e. $d \leq K$. That contradicts $d \geq K + M$, since $M \neq 0$. That contradiction proves $M = d$. That completes the proof of the lemma.

The \mathbf{d} -matrix equations simplify accordingly to

$$\begin{aligned} MN/K &= pa + Mb + ec \\ N - K^2 &= ga + fc \\ K^2 &= ha + rc \end{aligned} \tag{2}$$

Lemma 20 *Suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . Suppose also that K does not divide M^2 . If there is an N -tiling of ABC , then all the tiles on side AC have their β angle on AC with the β angle nearer to C than the other angle.*

Remark. Lemma 19 tells us that the tiles on AC have their a or c edges on AC , not their b edges. This lemma tells us exactly what orientation those tiles have.

Proof. By Lemma 19, we have $m = 0$, which means there are no b edges of tiles on AC . Hence the β angle of each tile with a side on AC occurs on AC . Since the two tiles at A both have their α angle at A , no β angle occurs at A . Hence the number of β angles of tiles on AC is equal to the number of vertices on AC that are not equal to A . By Lemma 5, exactly one β angle occurs at each vertex; so no two of the tiles on AC have their β angles at the same vertex. It follows, proceeding from A to C along AC , that each tile on AC has its β angle at the vertex nearer to C .

Next we have to prove that each tile sharing an edge with side BC has its α angle nearer to C than to B . The proof is similar to what we did in case $M = 1$. The tile at B has its β angle at B . Then the number of tiles sharing an edge with side X is the same as the number of vertices on side X , including C but not B . Since exactly one α angle occurs at each vertex on the interior of side X (by Lemma 5), proceeding from B towards C along X , we see by induction that each tile in turn has an angle at the corner nearest B that is not equal to α , and an α angle at the corner nearest C . (This is consistent with the result of Lemma 20, according to which the last tile has its α angle at C .) That completes the proof of the lemma.

Lemma 21 *Any interior segment with only b edges on one side, and at least one a or c on the other side, has length at least $Kb = K^2 - M^2$.*

Proof. Suppose L is an interior segment with only b edges on one side; say there are ℓ edges of length b on that side. On the other side there may be a , b , and c edges, so we have

$$\ell b = ua + vb + wc$$

for some nonnegative integers u , v , and w , and by hypothesis not both u and w are zero. Let $j = \ell - w$; then $jb = ua + vb$ and $j \neq 0$. By Lemma 18, we have $j \geq K$. The length of segment L is at least jb , which is at least Kb . Since $b = K - M^2/K$, we have $Kb = K^2 - M^2$. That completes the proof of the lemma.

Lemma 22 *Assume $jc = ua + vb + wc$, where a , b , and c are the lengths of the sides of a tile in an N -tiling with $N - M^2 = 2K^2$, and K does not divide M^2 , and j , u , v , and w are nonnegative integers with $0 < j < M$. Then $v = 0$.*

Proof. Without loss of generality we can assume $w = 0$ (by subtracting wc from both sides and replacing j by $j - w$). According to Lemma 18, which is applicable since $vb = jc - ua$, K divides v . Since $j < M$ and $c = K$ the left side of $jc = ua + vb$ is less than MK . Hence the right side is less than MK too; that is, $ua + vb < MK$. Hence $vb < MK$. Since $a < b$ we have $va < vb < MK$; since $a = M$ we have $vM < MK$, and dividing by M we have $v < K$. But since K divides v , we have $v = 0$. That completes the proof of the lemma.

Corollary 2 *If K does not divide M^2 , then an interior segment with only c edges on one side, and fewer than M of them, cannot have any b edges on the other side; and if $(K, M) = 1$ such an interior segment does not exist.*

Proof. Let j be the number of c edges on one side of the interior segment, and apply the lemma. That completes the proof.

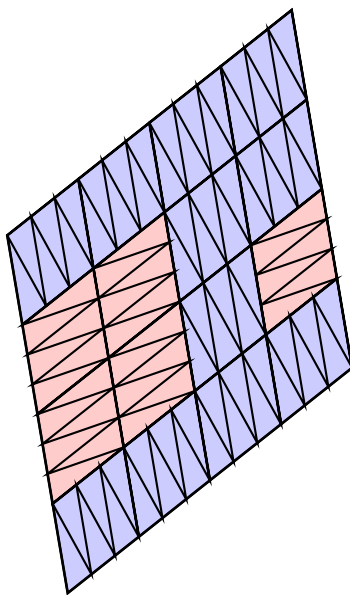
Remark. If $K = 12$, $M = 2$, $u = 1$, $v = j = 6$, we have $jK = uM + vb$, and K divides vM^2 but not M^2 .

13 The case when K does not divide M^2

We point out that it is possible for K and N to have a common factor, but still K does not divide M^2 . For example, when $M = 2$ and $K = 6$ we have $N = 2K^2 - M^2 = 68$, and neither condition holds.

We consider what a tiling would have to look like, starting from vertex B and proceeding downwards. The possible partial tilings can be complicated, because there is the possibility that some parallelograms within a partial quadratic tiling can be replaced by parallelograms tiled with tiles in another orientation. These tiles are in a different connected component of the graph. Fig. 7 gives a sample of what can happen. The colors indicate connected components.

Figure 7: A partial tiling when K and M are not relatively prime



The example in the figure is essentially an arbitrary tiling by copies of two different fundamental parallelograms tiled by triangles. The example could be made more complicated by shifting some of the parallelograms a bit; they do not have to line up checkerboard-style, as one can see from Fig. 6. But we will prove below that if you start tiling at the top vertex B (with the β angle) then you must get something that starts out like this. This kind of pattern must continue as you start from B and extend the tiling downwards, until you come to one of the vertices B or C . The important point is that only a and c edges will occur on AB and BC . But, in case K does not divide M^2 , we have proved that there must be at least M edges of length b on BC in any tiling of ABC by this tile. It will follow that no such tiling exists. The detailed proof follows.

Note that in the triquadratic tilings, there is a component boundary in Direction C that has a edges on one side, and b edges on the other. Under the present hypothesis that K does not divide M^2 , this is impossible because of Lemma 18. That is one of the main ideas of this proof. The other main idea is that there are no b edges on AB or AC .

For purposes of describing configurations, we think of ABC as oriented with B upwards, or to the “north”, and AC horizontal, with A at the “west” and C at the “east”. Also sometimes we use “above” and “below” as synonyms for “north of” and “south of”, and “right” and “left” as synonyms

for “east” and “west.” The “horizontal” is the direction of BC , and the “positive horizontal” is the direction of the ray BC .

Definition 5 A **center** of a tiling is a vertex at which four tiles meet, with three γ angles and a β angle.

The center vertex or vertices will play an important role in this section, as should not be surprising after seeing the pictures of the triquadratic tilings.

We start with some purely geometric lemmas, which will prove useful when we want to apply Lemma 18.

Lemma 23 Given triangle ABC with angle $A = 2\alpha$ and angle $B = \beta$ and $3\alpha + 2\beta = \pi$, the maximum length of a line parallel to the angle bisector of A is Kb . The maximum is taken on only when the line is the angle bisector of A and the other end of it is on BC .

Proof. Let PQ be a line in Direction A , with P on AB and Q on BC . All the triangles PBQ are similar as P varies, so the maximum length is taken on when $Q = A$. Similarly, if P lies on AC , the triangles PCQ are similar as P varies, and the maximum is taken on when $P = A$. In order to prove the first assertion of the theorem, then, we may assume without loss of generality that $P = A$. We must prove AQ has length Kb . Consider triangle ABQ ; this has angle α at A and angle β at B , so it is similar to the tile. Hence angle AQB is γ , and angle AQC is $\alpha + \beta$, the supplement of γ . But angle C is also $\alpha + \beta$, so triangle ABQ is isosceles. Hence the length of AQ is equal to the length of AC , which is $\lambda b = Kb$. That completes the proof of the lemma.

We next consider lines in Direction C , i.e. lines PQ with P on AB and Q on BC making an angle of β with the negative horizontal. The longest such line occurs when C lies on the line, and it has length less than Kb if and only if $\alpha < \beta$, as follows immediately from the observation that triangle APC is similar to triangle ABC , since the length of BC is Kb . shows. But in this paper we have not assumed $\alpha < \beta$, so such lines *can* be longer than Kb . Nevertheless the following lemma suits our purposes.

Lemma 24 Given triangle ABC with angle $A = 2\alpha$ and angle $B = \beta$ and $3\alpha + 2\beta = \pi$, let F be the point on the angle bisector of angle BAC such that angle $FCA = \beta$. Then the length of FC is less than Kb .

Proof. Triangle FCB is similar to the tile, since it has angle α at A and angle β at B . Then BC is opposite the γ angle of FCB , and FC is opposite the α angle. Since $\alpha < \gamma$, the length of FC is less than the length of BC , which is Kb by Lemma 1. That completes the proof of the lemma.

Lemma 25 Given triangle ABC with angle $A = 2\alpha$ and angle $B = \beta$ and $3\alpha + 2\beta = \pi$, let X be the point on BC where the angle bisector of angle BAC meets BC . Then the length of XC is Mb and the length of BX is Mc .

Proof. Triangle ABX is similar to the tile, since it has angle α at A and angle β at B . By Lemma 1, the length of AC is $\lambda c = Kc$. Therefore the length of BX , which is opposite the α angle of triangle ABX , is Ka . Since $a = M$ and $c = K$, this can also be written as Mc . According to Lemma 15, the length of BC is $M(b + c)$. The length of XC is obtained by subtracting the length of BX from the length of BC . We get $M(b + c) - Mc = Mb$ as claimed. That completes the proof of the lemma.

Lemma 26 *Given triangle ABC with angle $A = 2\alpha$ and angle $B = \beta$ and $3\alpha + 2\beta = \pi$, and given an N -tiling of ABC by the tile with angles α and β , let RW be a line segment in Direction C with R on AB and W on BC , and only c edges of tiles on BW . Suppose $N + M^2 = 2K^2$ is the solution of the tiling equation corresponding to the given tiling. Then the length of RW is at most Mb , which is less than Kb .*

Proof. In this paragraph we prove that the length of RW is less than Kb . Triangle BRW has angle β at B , and angle α at W , so it is similar to the tile. According to Lemma 15, there are exactly M edges of length b on BC . Since no b edges occur in the direction of BC in either a Type I or Type II component (they occur instead in directions A and C), the lowest possible position of W is on BC , at distance Mb from C . According to Lemma 15, the length of BC is $M(b + c)$. Therefore the length of BW is at most Mc . Since BW is opposite the γ angle of triangle RBW , the sides of RBW are larger than the corresponding sides of the tile by a factor of at most M . Since RW is opposite the β angle of triangle RBW , its length is less than or equal to Mb . Since $M < K$, the length of RW is less than Kb . That completes the proof of the lemma.

Recall from Definition 5 that a center of a tiling is a vertex where three γ angles and one β angle meet.

Lemma 27 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 , and there is an N -tiling of ABC . Let FH be a maximal segment in direction C , with F northwest of H , and Type I tiles northeast of FH , and only Type I or Type II tiles above FH extended have vertices at F or H . Then either there is a center at F , or all the tiles below FH with an edge on FH are also of Type I, i.e. have their b edges on FH and their α angles to the northwest*

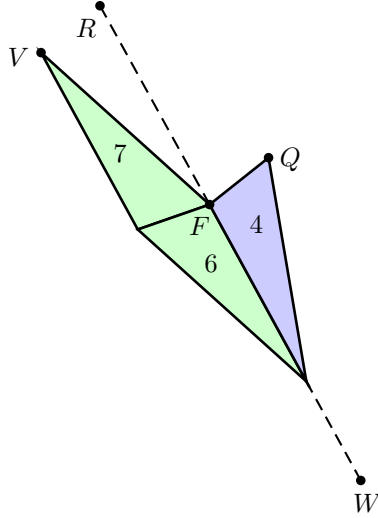
Proof. Let RW be the line segment containing FH , with R on AB and W on BC . Then either F lies on AB , or there is a Type II component to the west of F , or a Type I component that is out of sync with the one above FH . Similarly, either H lies on BC , or there is a Type II component to the east of H , or a Type I component that is out of sync with the Type I component above FH . Since there are no tile boundaries in Direction C in components of Type II, the components west of F or east of H do extend below RW at F or H (i.e., they extend below RW either exactly at F or H , or just west of F or just east of H).

Since the tiles above FH belong to components of Type I, they all have their b edges on RW . Since the component west of F extends below RW , or F is on AC , and the component east of H extends below RW , or H is on BC , there is a tile below FH with an edge on FH and a vertex at F , and a tile below FH with an edge on FH and a vertex at H . Since the length of RW is less than Kb , by Lemma 26, and FH lies on RW , the length of FH is less than Kb . Hence, by Lemma 18, all the tiles below FH with an edge on FH have their b edges on FH . The tiles above FH , being of Type I, have their α angles to the southeast and their γ angles to the northwest. Since $\gamma > \pi/2$, none of these vertices have two γ angles below FH . Therefore, by the pigeonhole principle, the tiles below FH with an edge on FH all have their γ angles in the same direction. Which direction is it?

Assume, for proof by contradiction, that F is not a center, and also that the tiles below FH with an edge on FH also have their γ angles to the northwest. Then at F there are two γ angles, one belong to Tile 4 above FH , and one belonging to Tile 6 below. Hence F does not lie on AB , since a boundary vertex has only one γ angle. Since F is not a center, there are no more γ angles at F , as three γ angles would require F to be a center. Hence there is another component \mathcal{D} to the northwest of F , and extending below FH at F ; \mathcal{D} is either of Type II or it is a Type I component, out of sync with the component \mathcal{C} of the tile at F with an edge on FH . The northwest edge of Tile

6 makes an angle of $\beta - (\pi - \gamma)$ with the negative horizontal, since RW makes an angle of β with the negative horizontal. That comes to $(\beta - (\alpha + \beta)) = -\alpha$. There are no tile boundaries in components of Type I or Type II in that direction, so Tile 6 is not on the boundary of \mathcal{D} . Let Tile 7 be the next tile, northwest of Tile 6, with a vertex at F . Since Tile 7 is not on the component boundary, Tile 7 is in the same component as Tile 6. Therefore Tile 7 it cannot share its b or its c side with the a side of Tile 6 (which forms the northwest boundary of Tile 6), by Lemma 16. Since it does not have its γ angle at F , Tile 7 must have its β angle at F . Let V be the northwest vertex of Tile 7; then VF makes an angle of $\beta - \alpha$ with the negative horizontal. See Fig. 8. But there are no tile boundaries of Type I or Type II tiles in that direction, since it is not parallel to AB or BC , and it is not Direction A or Direction C . Thus VF cannot be on the boundary of \mathcal{D} . But also VF *must* be on the boundary of \mathcal{D} , since there is otherwise no room for the boundary of \mathcal{D} to extend below RW , since angle VFR is α . This is a contradiction, completing the proof by contradiction started at the beginning of this paragraph, and hence also completing the proof of the lemma.

Figure 8: Both Tile 4 and Tile 6 have their γ angles at F .



Lemma 28 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 , and there is an N -tiling of ABC . Let FH be a maximal segment in direction A , with F southwest of H , and Type II tiles northeast of FH , and only Type I or Type II tiles above FH extended have vertices at F or H . Suppose it is not the case that $F = A$ and H lies on BC . Then either there is a center at H , or all the tiles below FH with an edge on FH are also of Type II, i.e. have their b edges on FH and their α angles to the southwest.*

Proof. Let ST be the line segment containing FH with S on AB and T on BC . We claim that the length of ST is less than or equal to Kb , and can be equal to Kb only if $S = A$. Triangle SBT is similar to the tile, since it has angle α at S , and angle β at B . By Lemma 1, the length of AC is Kc , so unless $S = A$, the length of BS is less than Kc . Since BS is opposite the γ angle of triangle SBT , and ST is opposite the β angle, the length of ST is less than Kb , unless $S = A$.

Therefore the length of FH is less than Kb , since we have excluded by hypothesis the case in which $F = A$ and H lies on BC . Since the direction of ST does not contain any tile boundaries of

tiles in Type I components, the Type I components that separate the segments FH do extend below ST , so there are tiles below FH at F and at H with vertices at F and H . Then by Lemma 18, all the tiles below FH with an edge on FH have their b edges on FH .

Let Tile 6 be the tile below FH at H with an edge on FH and a vertex at H , and Tile 4 the tile above FH with an edge on FH and a vertex at H . If Tile 6 does not have its γ angle at H , then by the pigeonhole principle and the fact that $\gamma > \pi/2$, we are finished. Hence we may assume that Tile 6 has its γ angle at H . Tile 4 also has its γ angle at H , since it is of Type II. Then H does not lie on BC , since there cannot be two γ angles at a boundary vertex. Hence there is a component \mathcal{D} east of H , which is either of Type I or is an out of sync Type II component, and \mathcal{D} extends below ST at H (or immediately beyond H). Then there is a point P below ST such that HP lies on the boundary of \mathcal{D} . The direction of HP cannot be parallel to AB or parallel to BC , as those directions lie in the interior of Tile 6. HP cannot lie in Direction A , as that would lie on ST ; the only possibility is that HP lies in Direction C , since the southern edge of Tile 6 is horizontal, so the northeastern edge of Tile 6 lies in Direction C . But in that case, component \mathcal{D} is right next to Tile 6, and it must be of Type I, not Type II, since tiles in a Type II component have no edges in Direction C . Let Tile 7 be the tile in component \mathcal{D} sharing the a edge of Tile 6. Since that edge is in Direction C , Tile 7 has its b edge against Tile 6. Since it is in a component of Type I, it has its γ angle at H . But we already know about two γ angles at H (from Tile 4 and Tile 6), so now we know there are three γ angles at H . Hence H is a center. That completes the proof of the lemma.

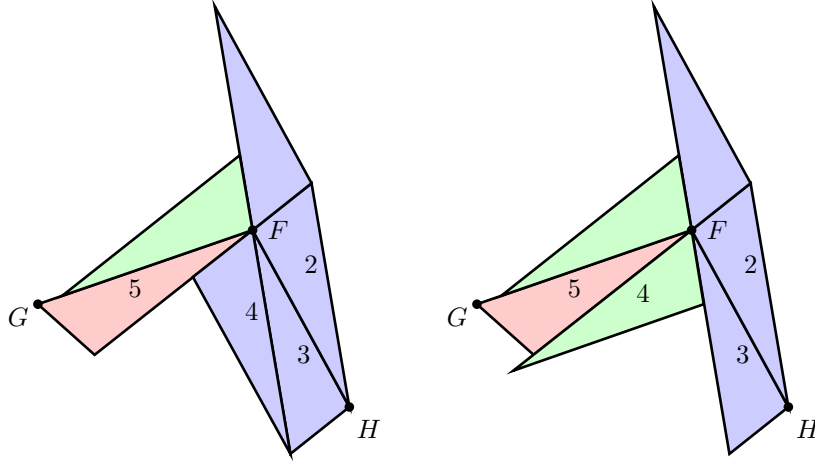
Lemma 29 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 , and there is an N -tiling of ABC . Let R and S be on AB and W and T be on BC , and suppose RW is in direction C and ST is in direction A , and suppose RW and ST intersect at F . Suppose that every tile that lies (wholly or partially) above SFW is of Type I or Type II. Then F is not a center.*

Proof. Assume, for proof by contradiction, F is a center. Then there is a type I tile northwest of F , with a vertex at F , sharing its b edge with FW , and a type II tile west of F , sharing its b edge with SF . Let E be the last point to the southwest of F such that there are only b edges of tiles on the north side of EF . Then southwest of that last tile, above EF , is a Type I component, which as we have seen extends below EF just to the southwest of E . Hence EF is a maximal segment. Hence all tiles below EF with an edge on EF have their b edges on EF . Hence the tile, say Tile 1, south of F with a vertex at F , has its b edge on EF . But now we argue similarly on the southeast of F : Let G be the last point on FW , southeast of F , such that there are only b edges on the north side of FG . Then southeast of G there is a Type II component, and FG is a maximal segment, so there are only b edges on the south side of FG . In particular, Tile 1 has its b edge on FG . But since $a \neq b$, Tile 1 cannot have a b edge on EF and also on FG . This contradiction completes the proof of the lemma.

There is another type of vertex, besides a center, that will be of interest in this section, and we now define it. Recall that “Direction A ” refers to the direction of the bisector of angle A , and “Direction C ” refers to the direction of the boundary between the tiles at vertex C , i.e., the direction making angle β with the negative horizontal.

Definition 6 *Vertex F is a **star** if there are vertices G and H such that GF and HF are tile edges in directions A and B respectively, and immediately above GHF there are only Type I and Type II tiles (so there are b edges on GF and FH), but some tile with a vertex at F is not of Type I or of Type II, and F is not a center.*

Figure 9: Two stars



Two stars are illustrated in Fig. 9. Note that the definition does not require that GH and FH be maximal segments—they might just be one tile edge long.

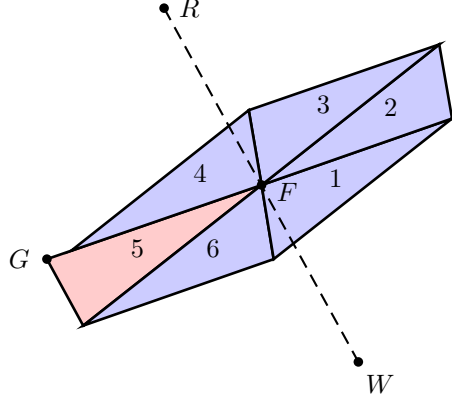
Lemma 30 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and there is an N -tiling of ABC . Let F be a star, and let GF and HG be in directions A and C , respectively. Let Tile 3 and Tile 5 be the tiles below HG and GF respectively, with a vertex at F . Then either Tile 3 has its a or c edge on HG , or Tile 5 has its a or c edge on GF .*

Proof. Suppose F is a star. Let Tile 3 be the tile below FH with an edge on FH . If Tile 3 has its a or c edge on FH , we are finished. Hence we may suppose that Tile 3 has its b edge on FH . Since there is a tile boundary in direction A , namely GF , the angle $GFH = \alpha + \beta$, so Tile 3 does not have its γ angle at F . Hence it has its α angle at F , so it is of Type I. This situation is illustrated in Fig. 9. Let P be the vertex of Tile 3 below FH . Let Tile 4 and Tile 5 be the two tiles filling angle GFP , with Tile 4 adjacent to Tile 3 and Tile 5 sharing an edge with GF . Tile 5 cannot have its α angle at F and its b side along FG , since in that case Tile 5 would be of type 2, and Tile 4 would have its β angle at F and its sides parallel to AB and BC , so it would be of Type I or Type II, and F would not be a star. Hence, if Tile 5 has its α angle at F , then it has its c side along GF , and we are finished. (The two stars in Fig. 9 illustrate the two possible positions of Tile 4 in this case.) Therefore we may suppose Tile 5 has its β angle at F . Then it has its c side or its a side on GF . That completes the proof of the lemma.

Definition 7 A *suspicious edge* is a tile boundary PQ in (the opposite of) Direction A or Direction C between two tiles that share a vertex P , and one of the tiles has a b edge along PQ and the other tile has an a or a c edge along PQ .

That is, the shared vertex P is at the northwest or northeast end of PQ . The location of the point Q is not important, so maybe we should have said “suspicious ray” instead of “suspicious edge”, but rays extend a long ways and we only care about the tiles with a vertex at P . Examples of suspicious edges occur at both centers and stars. The point of Lemma 30 is that there are suspicious edges at stars. There may be other types of vertices with suspicious edges as well. Fig. 10 illustrates a vertex with a suspicious edge, at which all but one tile are of Type II.

Figure 10: A vertex with five Type II tiles and one suspicious edge GF .



Lemma 31 Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 , and there is an N -tiling of ABC . Let R and S be on AB and W and T be on BC , and suppose RW is in direction C and ST is in direction A , and suppose RW and ST intersect at F . Suppose that every tile that lies (wholly or partially) above SFW is of Type I or Type II. Then F is not a star.

Proof. Suppose F is a star; then there are points G on SF and H on FW such that either GF or HF is a suspicious edge. Suppose first that GF is a suspicious edge. Then there is a tile, say Tile 1, above GF with a vertex at F , and its b edge on GF , while the tile, say Tile 2, below GF with a vertex at F , has its c or a edge on GF . Above ST there are only Type II and Type I tiles, and Type I tiles do not have any tile boundaries in Direction A , the direction of GF . Hence the maximal segment containing GF has only b edges on its northwest side. But by Lemma 23, its length is less than KB , since F does not lie on BC . But this contradicts Lemma 18. Hence GF cannot be a suspicious edge.

Therefore HF is a suspicious edge. Then there is a tile, say Tile 3, above HF with a vertex at F , and its b edge on HF , while the tile, say Tile 4, below HF with a vertex at F , has its c or a edge on HF . Above RW there are only Type II and Type I tiles, and Type II tiles do not have any tile boundaries in Direction C , the direction of HF . Hence the maximal segment containing HF has only b edges on its northeast side. Since only Type I and Type I tiles occur above RW , the point W lies on BC above the first b edge on BC . Let X be the point on BC where the angle bisector of angle BAC meets BC ; then by Lemma 25, the length of XC is Mb , so W lies on or above X . Let P be the point on AB such that PX is in Direction B ; that is, angle BXP is α . Then triangle BXP is similar to the tile, since it has angle β at B and angle α at X . By Lemma 25, the length of BX is Mc , so the similarity factor is M , since BX is opposite the γ angle of triangle BXP . Hence PX has length Mb . Since RW is parallel to PX and W is north of X , the length of RW is less than or equal to Mb , which is less than Kb since $M = a < c = K$. Hence the length of HF is less than Kb . But all the tiles above HF with their edges on HF are Type II tiles, and have their b edges on HF , while at least one tile below has its a or c edge on HF . That contradicts Lemma 18. That completes the proof of the lemma.

The tile at B must be of Type I or Type II, since it has sides parallel to AB and BC and its β angle at the top. We ask the reader to consider a line in Direction C that passes through the vertex B (hence missing the interior of ABC entirely) and to lower that line gradually, keeping it in

Direction C , until it first encounters a tile that is neither of Type I or Type II. What happens then is the subject of the next lemma.

Lemma 32 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 . Suppose there is an N -tiling of triangle ABC by the tile with angles α and β . Let RW be a line segment in direction C (and contained in the closed triangle ABC) with R northwest of W , and such that all tiles wholly or partially above RW are of Type I or Type II, but there is a point Q on RW , and strictly between R and W , that is on the boundary of a tile (below RW) that is neither of Type I nor of Type II. Suppose that the length of RW is less than Kb . Then either*

- (i) *Some such point Q is a center or a star at the northwest end F of a maximal segment FH contained in RW , or*
- (ii) *No such point Q touches any such segment FH , instead all such Q lie on the boundary of a Type II component with Q on FH (as illustrated in Fig. 10).*

In either case, there is a suspicious edge in direction A ending at Q .

Remarks. It is not required that R lie on AB and W lie on BC , although that is allowed, and if it is true, the hypothesis that the length of RW is less than Kb will be automatically satisfied. Also, the lemma does not rule out the existence of *other* such points Q , lying in the middle of a maximal segment FH with a star or center at the top.

Proof. First we prove the claim made in the last sentence, for stars and centers. By Lemma 30, stars have the requisite suspicious edge. Now suppose FH is a maximal segment in Direction C , lying on RW , and its northwest end F is a center. Since all the tiles above FH with an edge on FH have their b edges on FH , and since the length of RW is less than kB , all the tiles below FH with an edge on FH have their b edges on FH ; in particular the two tiles above and below FH at F have their γ angles at F and their b edges on FH . Let Tile 1 be the one below FH and Tile 2 the one above FH . Let Tile 3 be the tile with a vertex at F and sharing an edge with Tile 2, and let Tile 4 be the tile between Tile 1 and Tile 3. Then Tile 3 is at least partly above RW , so it is of Type I or Type 2. If Tile 3 has its γ angle at F , then its southwest edge makes an angle of 2γ with FH , which is to say an angle of $\beta - (2\gamma - \pi)$ with the horizontal. But

$$\begin{aligned} \beta - (2\gamma - \pi) &= \beta - 2(\beta + 2\alpha) + \pi \\ &= -\beta - 4\alpha + \pi \\ &= -\beta - 4\alpha + (2\beta + 3\alpha) \\ &= \beta - \alpha \end{aligned}$$

But Type I and Type II tiles have no edges at all in this direction. Hence Tile 4 does not have its γ angle at F .

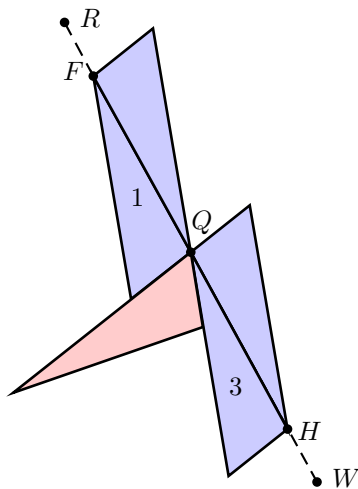
Hence Tile 3 has its β angle at F . Then Tile 4 is also partly above RW , so it too must be of Type I or Type II. It cannot be Type I, since Type I tiles have no edges in the direction (namely Direction A) of the boundary between Tile 4 and Tile 1. Therefore Tile 4 is of Type II. But Type II tiles have their b edges in Direction A , so we are finished: the required suspicious edge in direction A from the center F exists.

We now turn to the proof that either (i) or (ii) must hold. There are no tile boundaries in any component of Type II that lie in Direction C , but RW lies in Direction C . Therefore, all the tiles above RW that have an edge on RW belong to components of Type I. But there may be portions of RW that pass through Type II components, or out-of-sync Type I components (ones with no tile

edges on RW). We therefore consider maximal segments FH lying on RW , with F northwest of H , such that all the tiles above FH with an edge on FH are in Type I components. (There may be tiles between these Type I components that have only a vertex on FH .) A center cannot have a straight line through it lying on tile boundaries, so no center lies in the interior of segment FH . If F is a center, we have reached the conclusion of the lemma, so we may assume F is not a center. Then by Lemma 27, the tiles below FH all have their b edges on FH , their γ angles to the southeast, and their α angles to the northeast. Therefore they all belong to Type I components, and connect to the corresponding tiles above FH . Suppose that Q lies on FH (including possibly at H). We will show that the only possibility is that $Q = F$ and Q is a star. We distinguish the following cases:

Case 1, Q lies on the interior of FH . Then let Tile 1 and Tile 3 be the tiles below FH with an edge on FH to the northwest and southeast of Q , respectively. Then Tile 1 has its γ angle at Q and Tile 3 has its α angle at Q . Then only a β angle remains, and Tile 2 must have its β angle at Q and share a side each with Tile 1 and Tile 3.

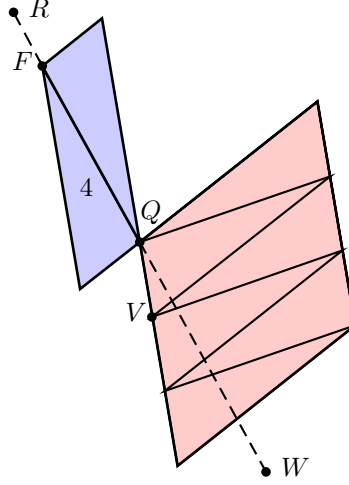
Figure 11: *Case 1.* The notch at Q must be filled with a Type I or Type II tile.



But the sides of Tile 1 and Tile 3 are parallel to AB and BC , so Tile 2 is of Type I or Type II, depending on whether its a side is against Tile 3 or Tile 1. See Fig. 11 for an illustration showing the notch at Q filled with a Type II tile. Hence Case 1 is ruled out.

Case 2, $Q = H$. Let Tile 4 be the tile below FH with its b side on FH and a vertex at H . Then Tile 4 has its γ angle at H . Let P be the southern vertex of Tile 4; then HP is parallel to AB . If H is on BC then Tile 2 has its β angle at H and sides parallel to AB and BC , so it is of Type I or Type II, contradiction. Hence H does not lie on BC . Hence there is a component \mathcal{D} of Type I or Type II east of H . Let HV be the first segment of the west boundary of \mathcal{D} , with V below RW . $HV = QV$ is not in Direction A , since then QV would lie in Tile 4. QV is not along $HP = QP$ (parallel to AB), since that would leave no room for Tile 2. QV is not in Direction C , since then it would lie on RW , but in fact it lies below RW . The only remaining possibility is that HV is parallel to BC , as shown in Fig. 12. But then angle PHV is β , and Tile 2 has a β angle at Q and sides parallel to AB and BC , so it is of Type I or Type II. Hence Case 2 is ruled out. (Although Fig. 12 illustrates the case when \mathcal{D} is of Type II, and has a vertex at Q , the argument also covers the cases when \mathcal{D} is an out-of-sync Type I component, or a Type 2 component with no vertex at Q .)

Figure 12: *Case 2.* When $Q = H$, there is still a notch with angle β to fill.



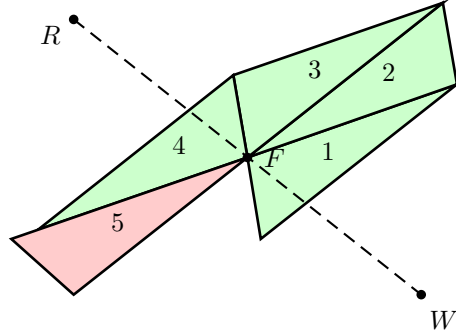
Case 3, $Q = F$. Let Tile 3 be the tile below FH with its b side on FH and a vertex at F . Let P be the vertex of Tile 3 that is not on FH . Tile 3 has its α angle at F , so FP is parallel to BC . Then if F lies on AB , Tile 2 must share an edge with Tile 3 and have its β angle at F , so it must be of Type I or II. Hence F does not lie on AB . Hence there is a component \mathcal{E} of Type I or Type II west of RW just north of F . Since \mathcal{E} extends below RW at F , there is a component boundary segment FG with G below RW . (We are not claiming that a tile in \mathcal{E} has a vertex at Q ; FG might be only part of an edge of a tile.) Suppose, for proof by contradiction, that the component \mathcal{E} is of type I (out of sync with the component above FH at F). Then its boundaries must lie in Direction C , or parallel to AB , or parallel to BC . Since FH is in Direction C , FG is not in that direction, since G lies below RW . If FG were parallel to BC it would lie on FP , and hence have a vertex at F , which it does not since it is out of sync with the component above FH . Hence FG must be parallel to AB . Then angle GFP is β , so there is only room for one tile in that angle, and it must be Tile 2. But then Tile 2 has two sides parallel to AB and BC respectively, and hence its component is of Type I or Type II, contradiction. Hence \mathcal{E} is not of Type I. Therefore \mathcal{E} is of Type II. We will show that F is a star; See Fig. 9 for an illustration. The boundary segment FG cannot lie on FP (as that would leave no room for Tile 2 between FP and \mathcal{E}), and it cannot be parallel to AB (which would make Tile 2 be of Type I or Type 2, having two edges parallel to AB and BC), so FG must be in direction A , the third boundary direction of Type II tiles. Tile 5 is not of Type II, since FG is a component boundary with Type II above it. Tile 5 is also not of Type I, since there are no tile boundaries of a type I tile in Direction A , which is the direction of FG . Hence F is a star; but in that case the conclusion of the theorem holds.

Now suppose that the point Q lies in or on the boundary of one of the components of Type II or out of sync Type I through whose closures RW passes.

Assume, for proof by contradiction, that Q lies inside an out of sync Type I component or on the boundary between two such components. (Here “out of sync” means that no tile edges occur on RW .) Since Tile 2, which is not of Type I or Type II, has a vertex at Q , and every tile even partially above RW is of Type I or Type II, it follows that the boundary between Tile 2 and the Type I component lies on RW . Hence, the Type I component in question cannot be out of sync after all. This contradiction shows that Q does not lie inside or on the boundary of a Type I segment.

Therefore, still assuming no center or star lies on RW , the point Q lies on RW and all tiles touching Q that lie partly above RW are of Type II. We fix Q to be the northernmost such point. We consider the configuration of tiles around Q . See Fig. 13. We have proved that Q does not touch

Figure 13: All tiles touching Q that lie partly above RW are of Type II.



any part of There are then four Type II tiles with a vertex at Q such tiles, shown as Tiles 1,2,3,4 in Fig. 10. The component that is not of type I or II must lie in the remaining $\alpha + \beta$ angle at Q , which lies below the line through Q parallel to ST . The four tiles just mentioned leave an angle $\alpha + \beta$ unfilled at Q . Let Tile 5 be adjacent to Tile 4. If Tile 5 has its β angle at Q , then it has its a or c side along Tile 4, and that edge is suspicious, as claimed. Hence we may assume Tile 5 has its α angle at Q , as shown in Fig. 10. Then Tile 6 has its β angle at Q , and its edges parallel to AB and AC , so it is of Type I or Type II. Since Q bounds some tile that is not of Type I or Type II, that tile must be Tile 5. If Tile 5 has its b edge along Tile 4, then since it has its α angle at Q , it is a Type II tile, contradiction. Hence the edge between Tile 4 and Tile 5 is suspicious, as claimed. That completes the proof of the lemma.

The next lemma is similar to the previous one, except that “Direction A” and “Direction C” are interchanged, and “Type I” and “Type II” are interchanged, and “northeast” and “northwest” are interchanged. Establishing more general terminology to make these two lemmas special cases of a single lemma is more trouble than repeating the proof with these changes, so here is the repetition. We do not, however, present new figures to illustrate the proof, as these would be similar to the figures above, but with the lines slanting down to the southwest instead of the southeast.

Lemma 33 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 . Suppose there is an N -tiling of triangle ABC by the tile with angles α and β . Let ST be a line segment in direction A (and contained in the closed triangle ABC) with S southwest of T , and such that all tiles above ST with an edge or vertex on ST are of Type I or Type II, but there is a point Q on ST , and strictly between S and T , that is on the boundary of a tile (below ST) that is neither of Type I nor of Type II. Suppose that the length of ST is less than Kb . Then either*

- (i) *Some such point Q is a center or a star at the northeast end H of a maximal segment FH contained in ST , or*
- (ii) *No such point Q touches any such segment FH , instead all such Q lie on the boundary of a Type I component with Q on ST .*

In either case, there is a suspicious edge in direction C ending at Q .

Proof. The line ST may pass through some Type I components; but the rest of it can be divided into maximal segments FH , such that all the tiles above FH with an edge on FH belong to Type II components, and hence have their b edges on FH . Fix such a maximal segment FH , with F southwest of H . The length of FH is less than Kb , since FH lies on ST which has length less than Kb . Since the direction of ST does not contain any tile boundaries of tiles in Type I components, the Type I components that separate the segments FH do extend below ST , so there are tiles below FH at F and at H with vertices at F and H . Then by Lemma 18, all the tiles below FH with an edge on FH have their b edges on FH .

Suppose H is a star. Then by Lemma 31, the requisite suspicious edge in Direction C exists. Suppose H is a center. Let Tile 1 be the tile above FH at H , and Tile 2 the tile below FH at H ; so both have their b edges on FH and their γ angles at H . Let Tile 3 be the tile adjacent to Tile 1 with a vertex at H . Then Tile 3 is of Type I or Type II. If it is of Type II, then it shares its a edge with Tile 1 and has its β angle at H , and the fourth tile with a vertex at H , say Tile 4, has its γ angle at H . Since it lies partly above ST , it must be of Type I or Type II. Both those types would have their β angles to the north (since the west edge is parallel to AB), so the b edge of Tile 4 is adjacent to the a edge of Tile 2, in Direction C as required. On the other hand, if Tile 3 is of Type I, then it has its c edge against Tile 1, and its α angle to the north (since the west edge is parallel to BC), so south edge of Tile 3 is parallel to AB . Hence the fourth tile, Tile 4, at H is partly above ST , so it is Type I or Type II. Since H is a center, Tile 4 has its γ angle at H . The edge between Tile 2 and Tile 4 is in Direction C and neither Type I nor Type II tiles have any a or c edges in that direction, so it must be the b edge of Tile 4 adjacent to Tile 2. Since Tile 2 has its a edge there, we have the required suspicious edge.

We may therefore assume, without loss of generality, that H is neither a star nor a center. Suppose that Tile 1 is below FH at H and Tile 2 above FH at H . Then Tile 2 is of Type II and has its γ angle at H . We claim that Tile 1 does not have its γ angle at H . In other words, if Tile 1 does have its γ angle at H , then H is either a star or a center. To prove this, suppose Tile 1 has its γ angle at H , but H is neither a star nor a center. Let the tiles be numbered clockwise around H from Tile 1, namely Tile 2, Tile 3, Tile 4, Tile 5, and Tile 6, and back to Tile 1. Then Tile 3 is of Type I or Type II. Suppose Tile 3 is of Type II. Then it shares its a edge with Tile 2 and has its β angle at H . Then Tile 4 cannot be of Type II, since if it were, it would have its α angle at H and an edge on FH extended, contradicting the fact that FH is a maximal segment. Since Tile 4 lies partly north of FH , it must be of Type I, so it has its c edge (parallel to BC) opposite its angle at H , and hence its γ angle at H , contradicting the fact that H is not a center. This contradiction shows that Tile 4 is not of Type I or Type II; but that is also a contradiction. This contradiction shows that Tile 3 is not of Type II. Since Tile 3 lies above FH , it is of Type I. Then it has its c edge (parallel to BC) adjacent to Tile 2, and its a edge parallel to BC has one end at H , so its β angle is at H . Tile 4 cannot have its α angle at H , as that would create a tile boundary extending FH , contradicting the fact that FH is a maximal segment. Hence Tile 4 is not of Type II. Tile 4 can also not have its γ angle at H as that would make H a center. Hence Tile 4 has its β angle at H . But then its b side, opposite the β angle, cannot be in Direction C , which it must be as Tile 4 must be of Type I, since it is partly above ST . This is a contradiction. This contradiction shows that Tile 3 cannot be of Type I, either; and this final contradiction completes the proof that Tile 1 does not have its γ angle at H .

Then, by the pigeonhole principle, all the tiles below FH with an edge on FH have their b edges on FH , with their α angles at the northeast and their γ angles at the southeast; so they are Type II tiles, and connect to the tiles above them.

We now distinguish three cases.

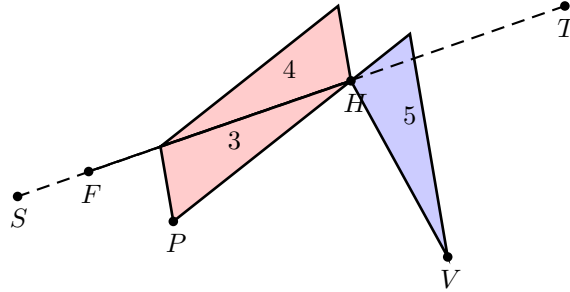
Case 1, the point Q lies in the interior of FH . Let Tile 8 and Tile 9 be the two tiles below

FH with an edge on FH and a vertex at Q , with Tile 8 on the southwest of S and Tile 9 on the northeast. Tile 2 is sandwiched between Tile 8 and Tile 9. Because the tiles below FH with an edge at FH have their α angles at the northeast, Tile 8 has its α angle at Q and Tile 9 has its γ angle at Q . Then Tile 2 must have its β angle at Q , and its sides are parallel to AB and BC , so it is after all of Type I or Type II, contradiction. Hence Case 1 is impossible.

Case 2, $Q = F$. Then F cannot be S , i.e. F cannot be on BC , since in that case Tile 2 has its β angle at F and sides parallel to AB and BC , so it is Type I or Type II, contradiction. Hence there is a component \mathcal{E} west of F , which is either of Type I or is an out of sync Type II component, and descends below ST . Let Tile 5 be the tile with its b edge on FH and a vertex at F . Suppose that Tile 5 does not have its γ angle at F . Then, since $\gamma > \pi/2$, the pigeonhole principle implies that each of the tiles below FH with an edge on FH has its γ angle to the northeast. In particular Tile 1, the tile under FH with a vertex at H , has its γ angle at H . But then, as proved above, H is a star or center; and that case has already been disposed of. Hence we may assume that Tile 5 has its γ angle at F . Then Tile 5 is of Type II, and the west edge of Tile 5 is parallel to BC , and the angle between it and SF is $\alpha + \beta$. Since F is an endpoint of the maximal segment FH , there is no tile boundary extending southwest from F in Direction A , so that direction from F lies in the interior of a tile, say Tile 3. Since Tile 3 lies partly above ST , it is of Type I or Type II; so its boundaries are either parallel to AB or BC , or in Direction A or Direction C . Let FG be the eastern edge of Tile 3. Then FG cannot be in Direction A since that lies on SF . It cannot be parallel to BC as that does not leave room for Tile 2 between Tile 5 and Tile 3. It cannot be in Direction C as that lies in the interior of Tile 5. The only remaining possibility is that FG is parallel to AB , i.e., angle $SFG = \alpha$. But then Tile 2 has side FG parallel to AB , and its east side (shared with Tile 1) parallel to BC , so its β angle is at F it must be of Type I or Type II, contradiction. This contradiction disposes of Case 2.

Case 3, $Q = H$. Let Tile 3 be the tile below FH at H with its b edge on FH and its γ angle at H . (See Fig. 14.) Let Tile 4 be the tile above FH at H , with its b edge on FH and its α angle

Figure 14: Case 3. $Q = H$. Then Q lies on the boundary of Type I tile 5.



at H . Let P be the southern vertex of Tile 3. Then PH is parallel to AB . H cannot be T , i.e. H does not lie on BC , since then Tile 2 (south of H , not shown in the figure) has a β angle at H and sides parallel to AB and BC , so it is of Type I or Type II, but by definition of Q it has some other type. Hence there is a component \mathcal{D} east of H , which is either of Type I or is an out of sync Type II component, and descends below ST . Let V be a point below ST such that HV is on the boundary of \mathcal{D} . Then HV is not parallel to AB , since that would put it on boundary of Tile 3 and leave no room for Tile 2. HV is not parallel to BC , since that would force Tile 2 to have a β angle at H and

sides parallel to AB and BC , so it would be of Type I or Type II. HV is not in Direction A , since then it would lie on ST . Therefore HV must lie in Direction C , and \mathcal{D} is a Type I component, since Type II components have no tile boundaries in Direction C . Then angle PHV is $\alpha + \beta$, which must be filled by two tiles, one of which is Tile 2. Let Tile 5 be the tile just north of HV . Then Tile 5 belongs to the Type I component \mathcal{D} , so it has its γ angle at H . Now Q is a star, since there are tile edges in both directions A and C from Q , and above $FHV = FQV$ are only Type I and Type II tiles, but below H there is a tile of another type. But we have shown we can assume that Q is not a star. That contradiction disposes of Case 3.

We now turn to the main theorem on the non-existence of tilings when K does not divide M^2 .

Theorem 3 *Suppose $3\alpha + 2\beta = \pi$, and α is not a rational multiple of π , and $N + M^2 = 2K^2$, and K does not divide M^2 . Then there is no N -tiling with $s = 2\sin(\alpha/2) = M/K$. In particular, given N , if there are no solutions of $N + M^2 = 2K^2$ such that K divides M^2 , then there are no N -tilings.*

Proof. Suppose, for proof by contradiction, that there is an N -tiling of ABC by the tile with angles α , β , and γ . Let ST be a line in direction A , with S on AB and T on BC , as low as possible so that all tiles wholly or partially above ST are of Type I or Type II. Let RW be a line in direction C , with R on AB and W on BC , as low as possible so that all tiles wholly or partially above RW are of Type I or Type II. Let F be the intersection point of RW and ST . By Lemma 31, F is not a star. By Lemma 29, F is not a center. By definition of RW , there is a point Q on RW such that some tile with a vertex at Q is neither of Type I nor Type II. Then Q cannot occur above F , since every tile above ST is of type I or type II. Then by Lemma 32, Q is on the boundary of a type II component, and there is a suspicious edge in Direction A ending at Q . Suppose, for proof by contradiction, that $Q = F$. Then the suspicious edge lies along SF ending at F . According to Lemma 33, since F is neither a center nor a star, F lies on the boundary of a Type I component, and does not touch any maximal segment on ST . But this is false, since the suspicious edge lies on ST . This contradiction shows that $Q \neq F$.

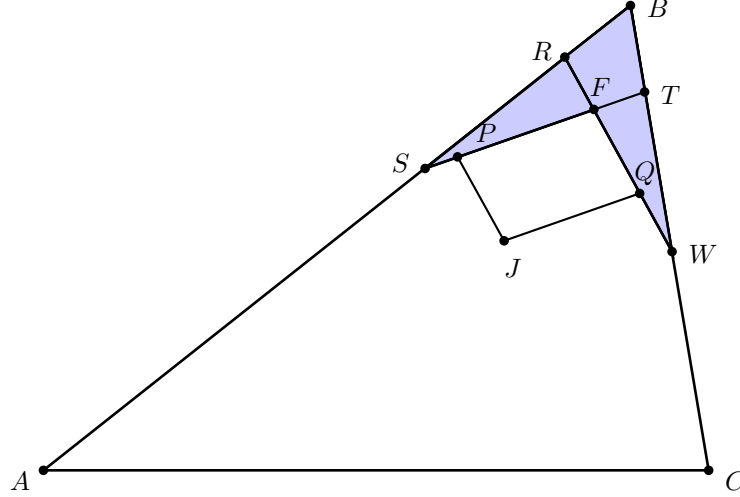
Since the tiles above RW are all of Type I or Type II, Q lies in the interior of FW , i.e., below F on RW . Similarly, using Lemma 33 instead of Lemma 32, there is a point P on ST below F , with a suspicious edge in direction C ending at P . See Fig. 15, which shows RW and ST intersecting at F , and the two suspicious edges emanating from Q and P , extended until they meet at a point J somewhere south of F . This forms a parallelogram $PFQJ$. We choose Q to be the northernmost such point on FW ; thus there are no points Q' between F and Q that lie on the boundary of a tile that is neither of Type I nor Type II, and such that there is a suspicious edge from Q' in direction A .

In Fig. 15, the shading above SFW indicates that only Type I and Type II tiles occur there; and although the lines SFW are shown in the figure, that does not imply that they consist of tile boundaries. However, in the vicinity of Q and P , there are tile boundaries along JQ and JP , where above JQ we have a tile with a b edge on QJ and a vertex at J , and below QJ , a tile with an a or c edge, and similarly on PJ at P . These tiles may or may not have edges on RW or ST , depending on whether Q is a star, a center, or something else.

Now consider other parallelograms $PF'HG$, where $F'H$ is in Direction C (hence parallel to FQ), and F' lies between P and F , and HG is in direction A (hence parallel to QJ and PF), and H lies inside the open parallelogram $PFQJ$, or $H' = Q$. such that, with Q' the intersection point of lines QJ and $F'H$, we have

- (i) All the tiles wholly or partially above $F'Q'$ (not just $F'H$) are of Type I or Type II, and
- (ii) There is a suspicious edge in direction A ending at H (i.e., extending southwest from H).

Figure 15: Above RFW we have Type I and Type II only. There are suspicious edges at P and Q .



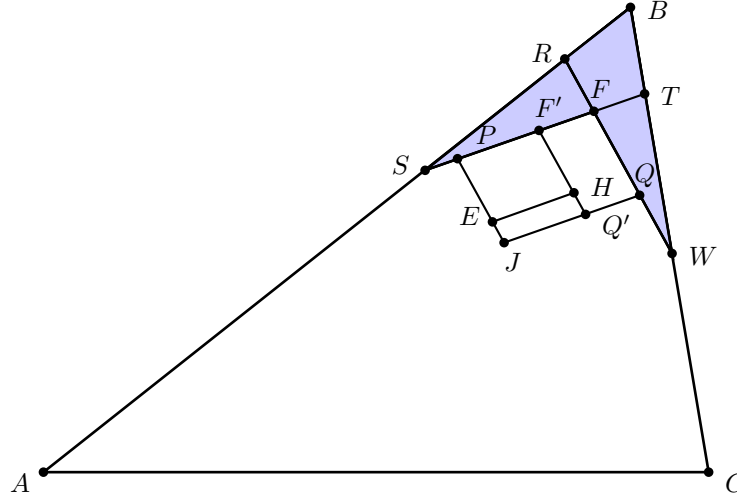
We have shown that there is at least one such parallelogram, namely the one with $F' = F$ and $H = Q$. For notational simplicity, we drop the primes, and assume that $FQJP$ is the smallest such parallelogram, that is, the one with F as close to P as possible. That means, in effect, that while we keep the assumption that tiles even partially above ST are of Type I or Type II, now we only assume that tiles (even partially) above RQ are of Type I or Type II, where we formerly had RW in place of RQ . We will derive a contradiction by constructing a smaller such parallelogram.

We begin by proving that there is some tile in the parallelogram $FQJP$ that is not of Type I or Type II. Suppose, for proof by contradiction, that there is no such tile. Since there are no tile boundaries in Direction A among Type I tiles, all the tiles in $FQJP$ with edges on JQ are of Type II, and since Type II tiles all have their b edges in Direction A , all the tiles in $FQJP$ with an edge on JQ have their b edges on JQ . But because a suspicious edge lies on JQ , ending at Q , there is at least one a or c edge on JQ belonging to a tile below JQ . Let M be a point on JQ southwest of Q , chosen so that MQ is the longest segment composed of tile boundaries in direction A containing point Q extending to the southwest from Q . At Q there are tiles both above and below MQ with a vertex at Q , by the definition of “suspicious edge”. Hence all the tiles above MQ with an edge on MQ have their b edge on MP . By Lemma 18, the length of MQ is less than Kb . But by Lemma 23, that implies that Q is the intersection point of BC and the angle bisector of angle BAC , and $M = A$. Then triangle ACQ is tiled, since the entire side AQ is composed of b edges of tiles. By Lemma 24, the length of QC is Mb . Since all the tiles above AQ are of Type I or Type II, they do not have their b edges on AQ . Since there are at least M edges of length b on BC , the entire segment QC must be composed of b edges. Then each tile with an edge on QC has a γ angle on QC . The bottom one, with a vertex at C , has its γ angle to the northwest, since angle C is less than γ . Since $\gamma > \pi/2$, no vertex on the boundary has more than one γ angle. By the pigeonhole principle, all the tiles with an edge on QC have their γ angles to the northeast. Let Tile 1 be the tile with an edge on BC below Q , and a vertex at Q . Then Tile 1 also has its γ angle to the northeast, so it cannot have any edge at all in direction A southeast of Q , let alone a suspicious edge. This contradiction shows that there is indeed a tile in the parallelogram $FQSP$ that is neither of Type I nor of Type II.

Now let us lower FQ . Specifically, let $F'Q'$ be the lowest line in Direction C (thus parallel to

FQ), with F' on PF and Q' on JQ , such that all the tiles above $F'Q'$ with an edge (or part of an edge, but more than a point) on $F'Q'$ are of Type I or Type II. Because of the existence of a tile in $FQJP$ that is neither of Type I nor of Type II, $F'Q'$ is still above PJ . But because we chose Q as far to the north as possible, Q' does not lie on FQ , i.e., $F'Q'$ definitely lies below FQ . See Fig. 16. Then all the tiles in the (nonempty) parallelogram $F'FQQ'$ are of Type I or Type II, since otherwise

Figure 16: Lowering FQ to $F'Q'$. Then H is the new Q .



$F'Q'$ would have been higher. Then $Q'Q$, being in Direction A , has only Type II tiles with edges on its upper side, since Type I tiles do not have any tile boundaries in Direction A . These edges must all be b edges, since Type II tiles have their b edges in Direction A . Since there is at least one a or c edge below $Q'Q$ (at Q), and since the length of $Q'Q$ is less than Kb , by Lemma 23, we see that Q' is not a vertex of a tile below $Q'Q$ with an edge on $Q'Q$, since if it were, Lemma 18 would imply a contradiction.

Let MP be, as above, the longest segment in direction A southwest of Q . Then M lies southwest of Q' , since otherwise MP has only b edges on its northwest side, but at least one a or c edge below it (at P), contradicting Lemma 18, since the length of JQ is less than Kb as noted above. Hence segment $Q'Q$ is composed of tile edges (and one part of a tile edge near Q'). Since $Q'Q$ is in Direction A , and the tiles above it are of Type I or Type II, the tiles above $Q'Q$ with an edge on $Q'Q$ are of Type II and have their b edges on $Q'Q$. There is a point H on $F'Q'$ such that H is a vertex of a tile that is neither of Type I nor of Type II, since otherwise $F'Q'$ would have been lower. Let E be a point on PJ such that EH is in direction A . We claim that $F'HEP$ is a smaller parallelogram than $FQJP$, but still satisfies conditions (i) and (ii).

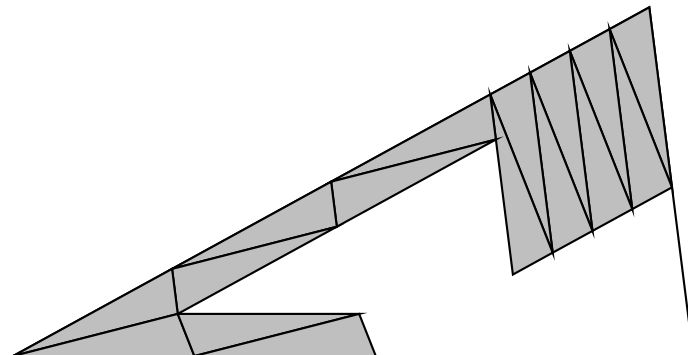
Condition (i) says that all the tiles wholly or partially above $F'Q'$ are of Type I or Type II. That is true by the definition of $F'Q'$.

The segment $F'Q'$ fulfills the hypotheses of Lemma 32, with (R, W, Q) in the lemma instantiated to (F', Q', H) . Hence there is a suspicious edge in direction A extending southwest from H . That is condition (ii). Hence our claim is proved: we have indeed constructed a smaller parallelogram than $FQJP$ that satisfies conditions (i) and (ii). But that contradicts the definition of $FQJP$ as the smallest such parallelogram. This contradiction completes the proof of the theorem.

14 Computer search for tilings

We wrote a computer program that searches for a boundary tiling of a given triangle by a given tile (under the hypotheses of this paper). Luckily, we later succeeded to find human-readable proofs of all the results originally obtained by the program, but still it is somewhat interesting. The program is about 1400 lines of C++. Its output is \TeX code that can be input to the `pstricks` graphics package.

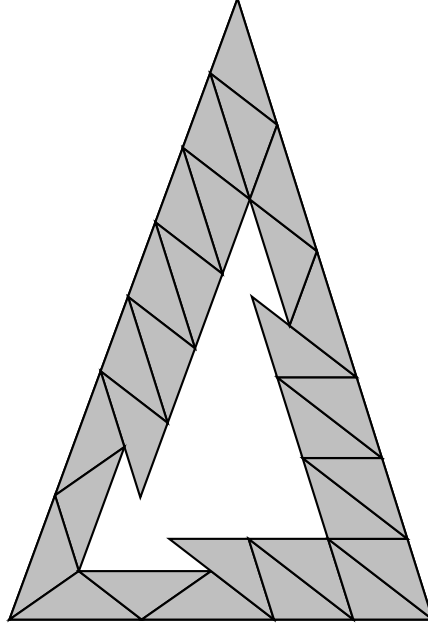
Figure 17: One of 30,836 failed attempts to find a 31-tiling



Consider, for example, the case $N = 41$. The equation $M^2 + 41 = 2K^2$ with $M^2 < 41$ has one (and only one) solution, namely $M = 3$ and $K = 5$. Then the tile must be $(MK, N - K^2, K^2)$, which is 15-16-25, and ABC must be $(MN, K(N - K^2), K^3)$, which is 123-80-125.

Since M and K are relatively prime, we know by Theorem 3 that there are no N -tilings. But before we knew that, we ran the computer program to search for boundary tilings. Since $N = 31$ had no boundary tilings, we thought perhaps $N = 41$ also had no boundary tilings. But computer search finds 34 boundary tilings, one of which is shown in Fig. 18. It is possible to examine the 34 boundary tilings and give simple reasons why each one cannot be filled in to construct a 41-tiling; so we knew before proving Theorem 3 that there were no 41-tilings. We still think it is of some interest that in some cases there are boundary tilings, even when there are no tilings, and in other cases there are not even boundary tilings.

Figure 18: Boundary tiling number 25 for $N = 41$.



15 Number theory of the tiling equation

Here we show that the tiling equation always has a solution when N has the right divisibility properties. Whether that solution corresponds to a tiling depends on whether K divides M^2 or not.

Lemma 34 *If the tiling equation $N = 2K^2 - M^2$ is solvable, then N is a square times a product of distinct primes, each of which is either 2 or is of the form $8n \pm 1$. If N is not a square or twice a square, then there is a solution with $0 < M < K$.*

Proof. Without loss of generality we can assume N is square free. If N is odd, then M is odd, so mod 8 the right side $2K^2 - M^2 = \pm 1$. Hence every odd prime dividing N is congruent to $\pm 1 \pmod{8}$, as claimed. If N is even then M is also even, so $N/2 = K^2 - 2(M/2)^2$ is congruent to $\pm 1 \pmod{8}$, and again every odd prime dividing N is congruent to ± 1 . That completes the proof.

Lemma 35 *Suppose N is a square times a product of distinct primes, each of which is either 2 or is of the form $8n \pm 1$. Then the tiling equation $N + M^2 = 2K^2$ has a solution in positive integers M and K with $M < K$.*

*Proof.*¹ Suppose N satisfies the stated divisibility conditions. We first prove that there exists an integer solution (M, K) . After that we will prove there is one with $0 < M < K$. The tiling equation asks for an integer in the field $\mathbb{Z}[\sqrt{2}]$ whose norm is $-N$. In detail, the integers of $\mathbb{Z}[\sqrt{2}]$ have the form $M + K\sqrt{2}$ and the norm of such an integer is $M^2 - 2K^2$. The standard theory of factorization in $\mathbb{Z}[\sqrt{2}]$ tells us that when N has the form given in the hypothesis, there is an integer solution. Specifically, because the norm is multiplicative, we can assume without loss of generality that N

¹The idea of this proof was given to me on MathOverflow by Noam Elkies and Will Jagy.

is an odd prime congruent to $\pm 1 \pmod 8$. (When $N = 2$ the equation is solvable with $M = 4$ and $K = 3$, and when $N = -1$ it is solvable with $M = 1$ and $K = 0$.) The form $x^2 - y^2$ has discriminant 8, so by Theorem 4.23 on p. 74 of [4], it integrally represents every prime congruent to $\pm 1 \pmod 8$. Hence the tiling equation does have some integer solution (M, K) .

It remains to prove that there is a solution with $M < K$. (Now we no longer assume N is prime.) One can verify by computation that if (M, K) is a solution of the tiling equation, then $(3M \pm 4K, 3K \pm 2M)$ is also a solution. One can verify it without computation by observing that this is the product of (M, K) with the unit $3 - 2\sqrt{2}$. Now choose a solution with M as small as possible but still positive. We may assume $K \geq 0$ since if (M, K) is a solution, so is $(M, -K)$. Since M is as small as possible, we have $3M - 4K \leq 0$. If $M < K$ we are done, so we may assume $K \leq M$. Then consider $M' = -(3M - 4K) = 4K - 3M$ and $K' = 3K + 2M$. Then (M', K') is also a solution, and $M' \geq 0$ and $K' > 0$. Then

$$\begin{aligned} K' - M' &= (3K + 2M) - (4K - 3M) \\ &= M - K \\ &\geq 0 \quad \text{since } K \leq M \end{aligned}$$

Hence $M' \leq K'$. But if $M' = K'$ then N is a square. If $M' = 0$ then $4K = 3M$, so $N = 2K^2 - M^2 = 2(3/4)^2 M^2 - M^2 = M^2/8 = 2(M/2)^2$. That completes the proof.

Remark. For large N , it will be easier to solve the tiling equation than to factor N .

16 The main theorems

The following theorem summarizes the results of this paper:

Theorem 4 *If the positive integer N is of the form $N = 2K^2 - M^2$ with $M^2 < N$ and K divides M^2 , then there is a (triquadratic) tiling of a certain triangle ABC , completely determined by M and K , using the tile T with integer sides $a = M$, $c = K$, and $b = K - M^2/K$. The angles of the tile are α and β where $2 \sin \alpha = M/K$ and $3\alpha + 3\beta = \pi$. The triangle ABC has angles 2α , β , and $\alpha + \beta$. No other triangle but this ABC can be N -tiled using T .*

If N does not have the specified form, or the tiling equation has no solution with $M^2 < N$, then there does not exist any N -tiling of any triangle by any tile with angles satisfying $3\alpha + 2\beta = \pi$.

Proof. Suppose, for proof by contradiction, that there is an N -tiling of ABC . By Lemma 4, there is only one possible shape for ABC : it has angles 2α , β , and $\beta + \alpha$. Since α is not a rational multiple of π (by hypothesis), we can apply Lemma 8 to conclude that $s = \sin(\alpha/2)$ satisfies a quadratic equation. Then s has to be rational, as we proved in [2], but here we can reach that conclusion without appealing to [2] by using Lemma 9, according to which $s = M/K$. We have

$$s^2 = \frac{2M^2}{M^2 + N}$$

Solving this for N we have

$$N = M^2 \left(\frac{2}{s^2} - 1 \right)$$

Setting $\lambda = M/s$ we have

$$N = \lambda^2 (2 - s^2)$$

as claimed. Odd primes not congruent to $\pm 1 \pmod 8$ divide N to an even power by Lemma 9. The existence of tilings when K divides M^2 , and the non-existence when K does not divide M^2 , are the main results above. That completes the proof.

Theorem 5 *Let N be squarefree, $N > 6$, and divisible by at least one prime congruent to $3 \pmod 4$. Then there are no N -tilings of any triangle by any tile, except possibly a tile with a 120° angle.*

Proof. Suppose N satisfies the given conditions. A squarefree number N is a sum of two squares if and only if all its odd prime divisors are congruent to $1 \pmod 4$, so our N is not a sum of two squares or twice a sum of two squares. Since it is squarefree, it is not of any of the forms n^2 , $2n^2$, $3n^2$, or $6n^2$, and hence there are no N -tilings of the forms considered in [1]. Hence, by the main theorem of [2], the only possible N -tilings use either a tile with $3\alpha + 2\beta = \pi$ and a triangle ABC of the shape considered in this paper, or a tile with a 120° angle. The latter are ruled out here by hypothesis. Suppose there is an N -tiling of the first kind. Then $N = 2K^2 - M^2$ with K dividing M^2 . Then the greatest common divisor $d = (K, M)$ is not 1; but then d^2 divides N , contradicting the assumption that N is squarefree. That completes the proof.

17 Progress on two problems of Erdős and one of Soifer

In [5] one finds two problems about triangle tiling posed by Erdős. We are now in a position to combine the results of this paper with those of [1] and [2] to reduce these problems to one unsolved case.

Soifer states (p. 48) the open problem solved in this paper, and says that Paul Erdős offered a \$25 prize for the first solution. He does not state where or when Erdős mentioned these problems. The problem statement is: Find all positive integers N such that at least one triangle can be cut into N triangles congruent to each other. In our terminology this asks, for which N can at least one triangle be N -tiled by some tile. This is Soifer's "Problem 6.7." The solution, we conjecture, is that N must be of one of the forms n^2 , $n^2 + m^2$, $2n^2$, $3n^2$, $6n^2$, or $2K^2 - M^2$ with $M^2 < N$ and K divides M^2 . We have proved that if N is not of one of those forms, there is no known N -tiling, and if there is any N -tiling at all, the tile has a 120° angle and the tiling satisfies other conditions given in [3]. The main theorem of [2] provides complete information characterizing the triples (N, ABC, T) such that ABC can be N -tiled by T , except in the case treated in this paper, where we also have characterized those triples; and except in the case when T has a 120° angle.

Soifer states that his "Problem 6.6" also a \$25 Erdős problem: Find (and classify) all triangles that can only be cut into n^2 congruent triangles for any integer n . We have reduced this problem, too, to the special case when the tile has a 120° angle. The reduction is given in the following theorem:

Theorem 6 *Triangles ABC admitting an N -tiling for some N that is not a perfect square, and in which the tile does not have a 120° angle, are exactly the following:*

- (i) *isosceles or equilateral triangles*
- (ii) *right triangles whose angles have rational tangents e/f where $e^2 + f^2$ is not a square.*
- (iii) *Triangles satisfying $2\alpha + 3\beta = \pi$ and α , where α is half of angle A , and α is not a rational multiple of π , and β is angle B , and $\sin(\alpha/2)$ is rational.*

Proof. If ABC is isosceles or equilateral then it has tilings when N is twice a square. If ABC is a right triangle with the tangents of its angles rational, let e and f be integers such that the tangent

of angle A is e/f , and for the tile choose a triangle similar to ABC , with angle α equal to angle a , and choose $c = \overline{BC}/e$ to be the hypotenuse of the tile, which determines $a = c \sin \alpha$ and $b = c \sin \beta$. Then ABC has a biquadratic tiling using this tile. The two half-tilings have a common component boundary along the altitude from B to AC , where f edges of length a meet e edges of length b . The total number of tiles is $N = e^2 + f^2$.

If ABC satisfies condition (iii), then we claim that for large enough N , there is an N -tiling by a (correspondingly small enough) tile with angles α and β as specified in (iii). We will choose a tile with sides a and c in the ratio $a/c = s = 2 \sin(\alpha/2)$. Since s is rational then there are integers m and k such that $a/c = M/K$. Then take $K = k^2$ and $M = mk$, and define $N = 2K^2 - M^2$. Since $a < c$, N is positive. There is an N -tiling of ABC by this tile if and only if K divides M^2 . But it does, since $M^2/K = (mk)^2/k^2 = m^2$.

We still have to check that $N = 2K^2 - M^2$ is not a perfect square.

Hence if triangle ABC satisfies (i), (ii), or (iii), then it admits an N -tiling for some N that is not a square. Conversely, if ABC is not of one of those forms, and N is arbitrary, and the tile does not have a 120° angle, then we proved in [2] that there is no N -tiling of ABC . That completes the proof.

Soifer also asks, in his “Problem 6.6”, whether we can count, i.e. predict, the number of ways of cutting ABC into N congruent triangles. That is, let $c(N, ABC)$ be the number of N tilings of ABC by any (all possible) tiles. He notes that this problem is more difficult than either of the two Erdős problems. We do not give a solution here, but we think the problem is within reach. We divide this problem into two pieces:

- (1) Given ABC and N , what are the possible tiles that can N -tile ABC ? and
- (2) Given ABC , N , and a tile T , how many N -tilings of ABC by T exist?

Problem 1 we have almost solved; it has been solved in [2] and this paper, except in the case of a tile with a 120° angle, which is partially solved in [3]. That leaves one unsolved case of (1), plus problem (2).

We think the notion of connected components will be useful to work on (2). We offer the following definition.

Definition 8 *Two tilings (using the same tile) are **immediately equivalent** if there is a parallelogram in a connected component of one tiling that can be replaced by a parallelogram lattice-tiled in a different way to obtain the second tiling. The relation of **equivalence** between tilings is the transitive closure of the relation of immediate equivalence.*

Note that two different tilings can only be equivalent if the sides a and b of the tile are commensurable, and if the scale is chosen so that they are integers, their least common multiple must be small enough that such a parallelogram exists. Here is a conjecture:

Conjecture 1 *Let ABC be N -tiled, and suppose $3\alpha + 2\beta = \pi$ and α is not a rational multiple of π . Then the tiling is equivalent to a triquadratic tiling.*

A similar conjecture cannot be true for all tilings. For example, we can start with an equilateral triangle ABC , cut it into four smaller equilateral triangles quadratically, and then tile each of those triangles equilaterally, reaching a 12-tiling of ABC . Alternately, we can first equilaterally tile ABC and then quadratically tile those three triangles, reaching another (different) 12-tiling of ABC . The first tiling has four connected components, the second has three. Neither contains any parallelograms so neither can be equivalent to any other tiling.

If the conjecture is true, we can count the number of tilings (when $3\alpha + 3\beta = \pi$) by counting the number of ways of picking disjoint parallelograms in the triquadratic tiling whose dimensions are multiples of the least common multiple of M and K .

References

- [1] Michael Beeson. Triangle tiling I: the tile is similar to ABC or has a right angle.

To appear, available on the author's website.

- [2] Michael Beeson. Triangle tiling II: Some non-existence theorems.

To appear, available on the author's website.

- [3] Michael Beeson. Triangle tiling IV: A non-isosceles tile with a 120 degree angle.

To appear, available on the author's website.

- [4] D. A. Buehl. *Binary Quadratic Forms*. Springer-Verlag, New York, 1989.

- [5] Alexander Soifer. *How Does One Cut a Triangle?* Springer, 2009.